A global theory of algebras of generalized functions II: tensor distributions

Michael Grosser, Michael Kunzinger,
Roland Steinbauer, James Vickers

Abstract

We extend the construction of [19] by introducing spaces of generalized tensor fields on smooth manifolds that possess optimal embedding and consistency properties with spaces of tensor distributions in the sense of L. Schwartz. We thereby obtain a universal algebra of generalized tensor fields canonically containing the space of distributional tensor fields. The canonical embedding of distributional tensor fields also commutes with the Lie derivative. This construction provides the basis for applications of algebras of generalized functions in nonlinear distributional geometry and, in particular, to the study of spacetimes of low differentiability in general relativity.

Keywords: Tensor distributions, algebras of generalized functions, generalized tensor fields, Schwartz impossibility result, diffeomorphism invariant Colombeau algebras, calculus in convenient vector spaces
MSC 2000: Primary 46F30; Secondary 46T30, 26E15, 58B10, 46A17

1 Introduction

The classical theory of distributions has long proved to be a powerful tool in the analysis of linear partial differential equations. The fact that there can in principle be no general multiplication of distributions ([37]), however, makes them of limited use in the context of nonlinear theories. On the other hand, in the early 1980's J.F. Colombeau ([4, 5, 6, 7]) constructed algebras of generalized functions $\mathcal{G}(\mathbb{R}^n)$ on Euclidean space, containing the vector space $\mathcal{D}'(\mathbb{R}^n)$ of distributions as a subspace and the space of smooth functions as a subalgebra. Colombeau algebras combine a maximum of favorable differential algebraic properties with a maximum of consistency properties with respect to classical analysis in the light of Laurent Schwartz' impossibility

result ([37]). They have since found diverse applications in analysis, in particular in linear and nonlinear PDE with non-smooth data or coefficients (cf., e.g., [34, 24, 33, 20, 12, 32, 10, 35] and references therein) and have increasingly been used in a geometrical context (e.g., [17, 27, 18, 19, 29, 30, 21]) and in general relativity (see e.g. [3, 1, 41, 28, 13] and [39] for a survey).

In this work we shall focus exclusively on so-called *full* Colombeau algebras which possess a canonical embedding of distributions. One drawback of the early approaches (given e.g. in [5]) was that they made explicit use of the linear structure of $\mathbb{R}^n$, obstructing the construction of an algebra of generalized functions on differentiable manifolds. This is in contrast to the situation with the so-called *special* algebras [18, Sec. 3.2] which are diffeomorphism invariant but do not allow a canonical embedding. It was only after a considerable effort that the *full* construction could be suitably modified to obtain diffeomorphism invariance: Building on earlier works of J.F. Colombeau and A. Meril ([8]) and J. Jelínek ([21]) a diffeomorphism invariant (full) Colombeau algebra $\mathcal{G}^d(\Omega)$ on open subsets $\Omega \subseteq \mathbb{R}^n$ was constructed in [17]. In this work a complete classification of full Colombeau-type algebras was given, resulting in two possible versions of the theory. In [22, 23], J. Jelínek was then able to prove that these algebras are, in fact, isomorphic, thereby providing a unique diffeomorphism invariant local theory. We will frequently refer to this construction as the "local theory". Finally, the construction of a full Colombeau algebra $\hat{\mathcal{G}}(M)$ on a manifold M based on intrinsically defined building blocks was given in [19]. Note that such an intrinsic construction is vital for applications in a geometric context: the two main fields of applications we have in mind are general relativity and Lie group analysis of differential equations. For applications in these fields, however, a theory of generalized tensor fields extending the above scalar construction is essential. In this paper we develop such a theory.

One might expect that going from generalized scalar fields to generalized tensor fields is straightforward and could be accomplished by considering generalized tensor fields as tensor fields with $\hat{\mathcal{G}}(M)$-functions as coefficients. However, the Schwartz impossibility result excludes such a construction as will be demonstrated in Section 4. More generally, we derive a Schwartz-type impossibility result for the tensorial case which applies to any natural (in the sense specified below) algebra of generalized functions.

To circumvent this road block we introduce an additional geometric structure into the theory which allows us to maintain the maximal possible differential algebraic properties and compatibility with the smooth case.

In more detail, the plan of this paper is as follows. We begin, in Section 2, by introducing some concepts and notation used throughout the paper. In Section 3 we present a new geometric approach to the scalar construc-

tion of [19] and point out some features which are essential in the context of the present work. In particular, we lay the foundations for establishing the impossibility results for the tensor case which are presented in Section 4. Section 5 exemplifies the guiding ideas of the tensorial theory by the special case of distributional vector fields and demonstrates the basic strategy for circumventing the no-go results alluded to above. Sections 6 and 8 form the core of our construction. The technically demanding proof of the fact that the embedded image of a distributional tensor field is smooth in the sense of [26] is given in Section 7. The concept of association—which provides 'backwards compatibility' of the new setting with the theory of distributional tensor fields—is the topic of Section 9. In the appendices we collect material on the key notion of transport operators (Appendix A) as well as some fundamental results on calculus in convenient vector spaces in the sense of [26] (Appendix B).

2 Notation

Here we fix some notation used throughout this article. I always stands for the interval $(0,1]$. Unless otherwise stated, M will denote an orientable, paracompact smooth Hausdorff manifold of (finite) dimension n. For subsets A, B of a topological space, we write $A \subset\subset B$ if A is a compact subset of the interior of B. Concerning locally convex vector spaces (which we always assume to be Hausdorff) we use the terminology and the results of [36]. In particular, "(F)-space" and "(F)-topology" abbreviate "Fréchet space" resp. "Fréchet topology". An (LF)-space is a strict inductive limit of an increasing sequence of (F)-spaces. A bornological isomorphism between locally convex spaces is a linear isomorphism respecting the families of bounded sets, in both directions. For details on the notion of smoothness in the sense of [26], see Appendix B.

For any vector bundle E over M, we denote by $\Gamma(M,E)$ resp. $\Gamma_{\mathrm{c}}(M,E)$ the linear spaces of smooth sections of E resp. of smooth sections of E having compact support. For $K \subset\subset M$, $\Gamma_{\mathrm{c},K}(M,E)$ stands for the subspace of $\Gamma_{\mathrm{c}}(M,E)$ consisting of all sections having their support contained in K. On $\Gamma(M,E)$, we consider the standard system of seminorms

$$p_{l,\Psi,L}(u) := \sum_{j=1}^{\dim E} \sup_{x\in L,|\nu|\le l} \left|\partial^\nu(\psi^j \circ (u|_V) \circ \psi^{-1}(x))\right|, \tag{2.1}$$

where $l \in \mathbb{N}_0$, (V,Ψ) is a vector bundle chart with component functions $\psi^1,\ldots,\psi^{\dim E}$ over some chart (V,ψ) on M and $L \subset\subset \psi(V)$ (cf. [18, p. 229]).

This leads to the usual (F)- resp. (LF)-topologies on $\Gamma(M, E)$ resp. $\Gamma_c(M, E)$ if M is separable (i.e., second countable). For general M, $\Gamma(M, E)$ becomes a product of (F)-spaces in this way, while the obvious inductive limit topology renders $\Gamma_c(M, E)$ a direct topological sum of (LF)-spaces. By a slight abuse of language, we will speak of (F)- resp. of (LF)-topologies also in the general case, being cautious when employing standard results on (F)- resp. (LF)-spaces. When there is no question as to the base space we will sometimes write $\Gamma(E)$ and $\Gamma_c(E)$ rather than $\Gamma(M, E)$ resp. $\Gamma_c(M, E)$. Finally, for an open subset U of the manifold M, we denote by $E|U$ the restriction of the bundle E to U. For some relevant basic facts on pullback bundles, two-point tensors and transport operators we refer to Appendix A.

Specializing to the tensor case, we denote by $\mathrm{T}^r_s M$ the bundle of (r, s)-tensors over M and by $\mathcal{T}^r_s(M)$ the linear space of smooth tensor fields of type (r, s). Also we write $\mathfrak{X}(M)$ resp. $\Omega^1(M)$ for the space of smooth vector fields resp. one-forms on M. By $\Omega^n_c(M)$ we denote the space of compactly supported (smooth) n-forms. The vector space of (scalar) distributions on M is then defined by $\mathcal{D}'(M) := (\Omega^n_c(M))'$. Following [31], we will view $\mathcal{D}'^r_s(M)$, the space of distributional tensor fields of type (r, s), as the dual space of tensor densities of type (s, r). Since the manifold M is orientable we may therefore write

$$\mathcal{D}'^r_s(M) := \Big(\mathcal{T}^s_r(M) \otimes_{\mathcal{C}^\infty(M)} \Omega^n_c(M)\Big)'.$$

We denote the action of the distributional tensor field $v \in \mathcal{D}'^r_s(M)$ on the (s, r)-density $\tilde{t} \otimes \omega$ by $\langle v, \tilde{t} \otimes \omega\rangle$.

Moreover, tensor distributions can be viewed as tensor fields with (scalar) distributional coefficients via the $\mathcal{C}^\infty(M)$-module isomorphism (cf., e.g., [18, Cor. 3.1.15])

$$\mathcal{D}'^r_s(M) \cong \mathcal{D}'(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^r_s(M). \tag{2.2}$$

We also mention the following useful representation of $\mathcal{D}'^r_s(M)$ as space of linear maps on dual tensor fields ([18, Th. 3.1.12]):

$$\mathcal{D}'^r_s(M) \cong \mathrm{L}_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{D}'(M)). \tag{2.3}$$

For the natural pullback action of a diffeomorphism μ on smooth or distributional sections of vector bundles we will write μ^*, the corresponding push-forward $(\mu^{-1})^*$ will be denoted by μ_*.

3 The scalar theory

To begin with we recall the following natural list of requirements for any algebra of generalized functions $\mathcal{A}(M)$ on a manifold M (cf. [17] for a full

discussion of the local case): $\mathcal{A}(M)$ should be an associative, commutative unital algebra satisfying

(i) There exists a linear embedding $\iota : \mathcal{D}'(M) \to \mathcal{A}(M)$ such that $\iota(1)$ is the unit in $\mathcal{A}(M)$.

(ii) For every smooth vector field $X \in \mathfrak{X}(M)$ there exists a Lie derivative $\hat{\mathrm{L}}_X : \mathcal{A}(M) \to \mathcal{A}(M)$ which is linear and satisfies the Leibniz rule.

(iii) ι commutes with Lie derivatives: $\iota(\mathrm{L}_X v) = \hat{\mathrm{L}}_X \iota(v)$ for all $v \in \mathcal{D}'(M)$ and all $X \in \mathfrak{X}(M)$.

(iv) The restriction of the product in $\mathcal{A}(M)$ to $\mathcal{C}^\infty(M)$ coincides with the pointwise product of functions: $\iota(f \cdot g) = \iota(f)\iota(g)$ for all $f, g \in \mathcal{C}^\infty(M)$.

In addition, for the purpose of utilizing such algebras of generalized functions in non-smooth differential geometry we will assume the following equivariance properties:

(v) There is a natural operation $\hat{\mu}^*$ of pullback under diffeomorphisms on $\mathcal{A}(M)$ that commutes with the embedding: $\iota(\mu^* v) = \hat{\mu}^*(\iota(v))$ for all $v \in \mathcal{D}'(M)$ and all diffeomorphisms $\mu : M \to M$.

Due to (iv), $\mathcal{A}(M)$ becomes a $\mathcal{C}^\infty(M)$-module by setting $f \cdot u := \iota(f)u$ for $f \in \mathcal{C}^\infty(M)$ and $u \in \mathcal{A}(M)$.

The celebrated impossibility result of L. Schwartz [37] states that there is no algebra $\mathcal{A}(M)$ satisfying (i)–(iii) and (iv'), where (iv') is a stronger version of (iv) in which one requires compatibility with the pointwise product of continuous (or $\mathcal{C}^k$, for some finite k) functions.

We now begin by recalling the construction of the intrinsic full Colombeau algebra $\hat{\mathcal{G}}(M)$ of generalized functions of [19] which possesses the distinguishing properties (i)–(v) above. We will put special emphasis on the geometric nature of the construction and point out the naturality of our definitions (see also [38])—as these are also essential features in the tensor case. The construction basically consists of the following two steps:

(A) Definition of a basic space $\hat{\mathcal{E}}(M)$ that is an algebra with unit, together with linear embeddings $\iota : \mathcal{D}'(M) \to \hat{\mathcal{E}}(M)$ and $\sigma : \mathcal{C}^\infty(M) \to \hat{\mathcal{E}}(M)$ where σ is an algebra homomorphism and both σ and ι commute with the action of diffeomorphisms. Definition of Lie derivatives $\hat{\mathrm{L}}_X$ on $\hat{\mathcal{E}}(M)$ that coincide with the usual Lie derivatives on $\mathcal{D}'(M)$ (via ι) resp. on $\mathcal{C}^\infty(M)$ (via σ).

(B) Definition of the spaces $\hat{\mathcal{E}}_{\mathrm{m}}(M)$ of moderate and $\hat{\mathcal{N}}(M)$ of negligible elements of the basic space $\hat{\mathcal{E}}(M)$ such that $\hat{\mathcal{E}}_{\mathrm{m}}(M)$ is a subalgebra of $\hat{\mathcal{E}}(M)$ and $\hat{\mathcal{N}}(M)$ is an ideal in $\hat{\mathcal{E}}_{\mathrm{m}}(M)$ containing $(\iota - \sigma)(\mathcal{C}^\infty(M))$. Definition of the algebra as the quotient $\hat{\mathcal{G}}(M) := \hat{\mathcal{E}}_{\mathrm{m}}(M)/\hat{\mathcal{N}}(M)$.

Observe that step (A) serves to implement properties (i)–(iii) and (v) of the above list while step (B) guarantees the validity of (iv). Since step (A) describes the basic space underlying our construction of generalized functions we refer to this step (by analogy with analytic mechanics) as giving the *"kinematics"* of the construction, and since step (B) refers to additional (asymptotic) conditions which we impose on the objects, we will refer to this step as giving the *"dynamics"* of the construction.

To introduce the kinematics part of the theory we discuss the question of the embeddings which will lead us to a natural choice of the basic space. We wish to embed both the space of smooth functions $C^\infty(M)$ and the space of distributions $\mathcal{D}'(M)$. Since smooth functions depend upon points $p \in M$ and distributions depend upon compactly supported n-forms it is natural to take our space of generalized functions to depend upon both of these. However, for technical reasons it is convenient to only use normalized n-forms.

Definition 3.1.

(i) The space of compactly supported n-forms with unit integral is denoted by

$$\hat{\mathcal{A}}_0(M) := \{\omega \in \Omega^n_{\mathrm{c}}(M) : \int_M \omega = 1\}.$$

(ii) The basic space of generalized scalar fields is given by

$$\hat{\mathcal{E}}(M) := \mathcal{C}^\infty(\hat{\mathcal{A}}_0(M) \times M).$$

Here and throughout this paper, smoothness is understood in the sense of calculus in convenient vector spaces ([26]), which provides a natural and powerful setting for infinite-dimensional global analysis. A map between locally convex spaces is defined to be smooth if it maps smooth curves to smooth curves. For some facts on convenient calculus in the context of the scalar theory we refer to [17, Sec. 4]. More specific results pertaining to the present paper are developed in Appendix B. Elements of the basic space will be denoted by R and their arguments by ω and p.

Definition 3.2. We define the embedding of smooth functions resp. distributions into the basic space by

$$\sigma(f)(\omega, p) := f(p) \quad \text{and} \quad \iota(v)(\omega, p) := \langle v, \omega \rangle.$$

Note that we clearly have $\sigma(fg) = \sigma(f)\sigma(g)$.

The second ingredient of the kinematics part of the construction is the definition of an appropriate Lie derivative. Given a complete vector field X, the Lie derivative of a geometric object defined on a natural bundle on a manifold M may be given in terms of the pullback of the induced flow (Appendix A and [25]). This geometric approach has the further advantage that in every instance the Leibniz rule is an immediate consequence of the chain rule. In order to define the Lie derivative of an element $R \in \hat{\mathcal{E}}(M)$ we therefore first need to specify the action of diffeomorphisms on $\hat{\mathcal{E}}(M)$.

Given a diffeomorphism $\mu : M \to M$ we have the following pullback actions of μ on the spaces of smooth functions resp. of distributions:

$$\mu^* f(p) := f(\mu p) \quad \text{and} \quad \langle \mu^* v, \omega \rangle := \langle v, \mu_* \omega \rangle,$$

where $\mu p := \mu(p)$ and $\mu_* \omega$ denotes the push-forward of the n-form ω. Hence the natural choice of definitions is the following.

Definition 3.3.

(i) The action of a diffeomorphism μ of M on elements of $\hat{\mathcal{E}}(M)$ is given by
$$(\hat{\mu}^* R)(\omega, p) := R(\mu_* \omega, \mu p).$$

(ii) The Lie derivative on $\hat{\mathcal{E}}(M)$ with respect to a complete smooth vector field X on M is
$$\hat{\mathrm{L}}_X R := \left.\frac{d}{d\tau}\right|_{\tau=0} (\widehat{\mathrm{Fl}_\tau^X})^* R,$$
where Fl_τ^X denotes the flow induced by X at time τ.

It is now readily shown that

$$\hat{\mu}^* \circ \sigma = \sigma \circ \mu^* \quad \text{and} \quad \hat{\mu}^* \circ \iota = \iota \circ \mu^*$$

which immediately implies

$$\hat{\mathrm{L}}_X \circ \sigma = \sigma \circ \mathrm{L}_X \quad \text{and} \quad \hat{\mathrm{L}}_X \circ \iota = \iota \circ \mathrm{L}_X.$$

Moreover, an explicit calculation gives

$$\hat{\mathrm{L}}_X R(\omega, p) = -d_1 R(\omega, p)\, \mathrm{L}_X \omega + \mathrm{L}_X R(\omega, .)\,|_p$$

which is precisely the definition of the Lie derivative in the general case given in equation (14) of [19].

Having established (i)–(iii) and (v) we now turn to step (B), i.e., the dynamics part of our construction. The key idea in establishing (iv) is to identify, via a quotient construction, the images of smooth functions under both the embeddings: For smooth f one has $\sigma(f)(\omega, p) = f(p)$, whereas regarding f as a distribution, one has $\iota(f)(\omega, p) = \int f(q)\omega(q)$. In order to identify these two expressions we would like to set $\omega(q) = \delta_p(q)$. Clearly this is not possible in a strict sense, but replacing the n-form ω by a net of n-forms $\Phi(\varepsilon, p)$ which tend to δ_p appropriately as $\varepsilon \to 0$ and using suitable asymptotic estimates shows the right way to proceed.

We begin by defining an appropriate space of delta nets (see [19] for details).

Definition 3.4.

(1) An element $\Phi \in \mathcal{C}^\infty(I \times M, \hat{\mathcal{A}}_0(M))$ is called a smoothing kernel if it satisfies the following conditions

  (i) $\forall K \subset\subset M\ \exists\, \varepsilon_0, C > 0\ \forall p \in K\ \forall \varepsilon \leq \varepsilon_0$: $\operatorname{supp} \Phi(\varepsilon, p) \subseteq B_{\varepsilon C}(p)$

  (ii) $\forall K \subset\subset M\ \forall k, l \in \mathbb{N}_0\ \forall X_1, \ldots, X_k, Y_1, \ldots, Y_l \in \mathfrak{X}(M)$

$$\sup_{\substack{p \in K \\ q \in M}} \|\mathrm{L}_{Y_1} \ldots \mathrm{L}_{Y_l}(\mathrm{L}'_{X_1} + \mathrm{L}_{X_1}) \ldots (\mathrm{L}'_{X_k} + \mathrm{L}_{X_k})\Phi(\varepsilon, p)(q)\| = O(\varepsilon^{-(n+l)})$$

where L'_X is the Lie derivative of the map $p \mapsto \Phi(\varepsilon, p)(q)$ and L_X is the Lie derivative of the map $q \mapsto \Phi(\varepsilon, p)(q)$. The space of smoothing kernels on M is denoted by $\tilde{\mathcal{A}}_0(M)$. We will use the notations $\Phi(\varepsilon, p)$ and $\Phi_{\varepsilon,p}$ interchangeably.

(2) For each $m \in \mathbb{N}$ we denote by $\tilde{\mathcal{A}}_m(M)$ the set of all $\Phi \in \tilde{\mathcal{A}}_0(M)$ such that for all $f \in \mathcal{C}^\infty(M)$ and all $K \subset\subset M$

$$\sup_{p \in K} \left| f(p) - \int_M f(q)\Phi(\varepsilon, p)(q) \right| = O(\varepsilon^{m+1})$$

The norms and metric balls in this definition are to be understood with respect to some Riemannian metric, but the asymptotic estimates are independent of the choice of metric.

We may now define the subspaces of moderate and negligible elements of $\hat{\mathcal{E}}(M)$ and carry out the announced quotient construction.

Definition 3.5.

(i) $R \in \hat{\mathcal{E}}(M)$ is called moderate if
$\forall K \subset\subset M\ \forall k \in \mathbb{N}_0\ \exists N \in \mathbb{N}\ \forall\ X_1, \dots, X_k \in \mathfrak{X}(M)\ \forall\ \Phi \in \tilde{\mathcal{A}}_0(M)$

$$\sup_{p \in K} |\mathrm{L}_{X_1} \dots \mathrm{L}_{X_k}(R(\Phi(\varepsilon, p), p))| = O(\varepsilon^{-N}).$$

The subset of moderate elements of $\hat{\mathcal{E}}(M)$ is denoted by $\hat{\mathcal{E}}_{\mathrm{m}}(M)$.

(ii) $R \in \hat{\mathcal{E}}_{\mathrm{m}}(M)$ is called negligible if
$\forall K \subset\subset M\ \forall k, l \in \mathbb{N}_0\ \exists m \in \mathbb{N}\ \forall\ X_1, \dots, X_k \in \mathfrak{X}(M)\ \forall \Phi \in \tilde{\mathcal{A}}_m(M)$

$$\sup_{p \in K} |\mathrm{L}_{X_1} \dots \mathrm{L}_{X_k}(R(\Phi(\varepsilon, p), p))| = O(\varepsilon^{l}).$$

The subset of negligible elements of $\hat{\mathcal{E}}_{\mathrm{m}}(M)$ is denoted by $\hat{\mathcal{N}}(M)$.

(iii) The Colombeau algebra of generalized functions on M is defined by

$$\hat{\mathcal{G}}(M) := \hat{\mathcal{E}}_{\mathrm{m}}(M)/\hat{\mathcal{N}}(M).$$

One now proves that $(\iota - \sigma)(\mathcal{C}^\infty(M)) \in \hat{\mathcal{N}}(M)$ by recourse to the local theory ([17]). So we obtain (iv) and since the properties obtained in step (A) are not lost in the quotient construction we indeed have (i)–(v). Note, however, the following subtlety: The fact that $\hat{\mathcal{G}}(M)$ is a *differential algebra* depends on the invariance of the tests for moderateness and negligibility under the action of the generalized Lie derivative $\hat{\mathrm{L}}_X$. This, however, is surprisingly hard to prove and has been done in [19] by recourse to the local theory as well.

We conclude this section with a lemma which will turn out to be useful for proving the analogue of $(\iota - \sigma)(\mathcal{C}^\infty(M)) \in \hat{\mathcal{N}}(M)$ in the tensor case (Theorem 8.12 (iii)).

Lemma 3.6. *Let $g \in \mathcal{C}^\infty(M \times M)$ satisfy $g(p,p) = 0$ for all $p \in M$, and let $m \in \mathbb{N}_0$. Then for every $\Phi \in \tilde{\mathcal{A}}_m(M)$ and every $K \subset\subset M$ we have*

$$\sup_{p \in K} \left| \int_M g(p,q) \Phi(\varepsilon, p)(q) \right| = O(\varepsilon^{m+1}). \tag{3.1}$$

Proof. Without loss of generality we may assume that K is contained in some open set W where (W, ψ) is a chart on M. Fixing L such that $K \subset\subset L \subset\subset W$ there is an $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ and all $p \in K$ we have $\operatorname{supp} \Phi(\varepsilon, p) \subseteq L$, by (1)(i) of Definition 3.4. Hence the integral in (3.1) may be written in local coordinates as

$$\int_{\psi(W)} \tilde{g}(x,y) \varepsilon^{-n} \phi(\varepsilon, x) \left(\tfrac{y-x}{\varepsilon}\right) \mathrm{d}^n y$$

where $\tilde{g} = g \circ (\psi \times \psi)^{-1} \in \mathcal{C}^\infty(\psi(W) \times \psi(W))$ and $\phi : D\,(\subseteq I \times \psi(W)) \to \mathcal{A}_0(\mathbb{R}^n)$ has the properties specified in [19, Lemma 4.2 (A)(i)(ii)]. In particular, D contains $(0, \varepsilon_1] \times \psi(K)$ for some $\varepsilon_1 \leq \varepsilon_0$ in its interior and we have $\sup_{x \in K'} |\int_{\mathbb{R}^n} \phi(\varepsilon, x)(y) y^\beta \mathrm{d}^n y| = O(\varepsilon^{m+1-|\beta|})$ for all multiindices β with $1 \leq |\beta| \leq m$ and all $K' \subset\subset \psi(W)$. Now a Taylor argument (analogous to the one in the proof of [17, Th. 7.4 (iii)], with α set equal to 0) establishes (3.1). $\square$

Note that for $g(p, q) = f(p) - f(q)$ where $f \in \mathcal{C}^\infty(M)$, the asymptotic estimate (3.1) is nothing but the condition defining the space $\tilde{\mathcal{A}}_m(M)$.

4 No-Go results in the tensorial setting

In this section we establish some general no-go results in the spirit of the Schwartz impossibility theorem [37], valid for tensorial extensions of *any* algebra $\mathcal{A}(M)$ of generalized functions satisfying the set of requirements stated in Section 3. For a comprehensive discussion tailored to the special case $\mathcal{A}(M) = \hat{\mathcal{G}}(M)$ we refer to [16].

Throughout this section we suppose that $\mathcal{A}(M)$ is any associative, commutative unital algebra with embedding $\iota : \mathcal{D}'(M) \to \mathcal{A}(M)$ satisfying conditions (i)–(v) from Section 3.

We first note that such an ι cannot be $\mathcal{C}^\infty(M)$-linear. In fact, let $M = \mathbb{R}$. Then supposing that ι is $\mathcal{C}^\infty(\mathbb{R})$-linear we derive the following contradiction:

$$\iota(\delta) = \iota(1)\iota(\delta) = \iota(v.p.\frac{1}{x} \cdot x)\iota(\delta) = \iota(v.p.\frac{1}{x})\iota(x\delta) = 0.$$

Clearly this calculation can be pulled back to any manifold. Thus, in general,

$$\iota(fv) \neq \iota(f) \cdot \iota(v) \qquad (f \in \mathcal{C}^\infty(M),\ v \in \mathcal{D}'(M)), \tag{4.1}$$

or, $\iota(fv) \neq f \cdot \iota(v)$, for any algebra $\mathcal{A}(M)$ of generalized functions as above.

As we shall demonstrate, this basic observation forecloses the most obvious way of extending a given scalar theory of algebras of generalized functions to the tensorial setting.

To this end, we write the natural embedding $\rho^r_s : \mathcal{T}^r_s(M) \to \mathcal{D}'^r_s(M)$ given by

$$\langle \rho^r_s(t), \tilde{t} \otimes \omega \rangle := \int_M (t \cdot \tilde{t})\, \omega \qquad (t \in \mathcal{T}^r_s(M),\ \tilde{t} \in \mathcal{T}^s_r(M),\ \omega \in \Omega^n_c(M))$$

in a different manner: Recall from (2.2) that

$$\mathcal{D}'^r_s(M) \cong \mathcal{D}'(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^r_s(M).$$

Denoting by ρ the standard embedding of $\mathcal{C}^\infty(M)$ into $\mathcal{D}'(M)$, the fact that ρ is $\mathcal{C}^\infty(M)$-linear allows one to rewrite ρ^r_s as

$$\rho^r_s = \rho \otimes_{\mathcal{C}^\infty(M)} \mathrm{id} : \mathcal{C}^\infty(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^r_s(M) \to \mathcal{D}'(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^r_s(M). \quad (4.2)$$

Given $\mathcal{A}(M)$ as above it is therefore natural to define the space of tensor-valued generalized functions as the $\mathcal{C}^\infty(M)$-module of tensor fields with generalized coefficients from $\mathcal{A}(M)$, i.e.,

$$\mathcal{A}^r_s(M) := \mathcal{A}(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^r_s(M). \quad (4.3)$$

It is then tempting to mimic (4.2) and define an embedding of $\mathcal{D}'^r_s(M)$ into $\mathcal{A}^r_s(M)$ by

$$\iota \otimes \mathrm{id} : \mathcal{D}'(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^r_s(M) \to \mathcal{A}(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^r_s(M). \quad (4.4)$$

The following result, however, shows that this map is not well-defined (not even in the scalar case $r = s = 0$) and therefore cannot serve as the desired embedding of $\mathcal{D}'^r_s(M)$ into $\mathcal{A}^r_s(M)$:

Proposition 4.1. *For any algebra $\mathcal{A}(M)$ which satisfies (i) from Section 3 and is a unital $\mathcal{C}^\infty(M)$-module, the following are equivalent:*

(i) $\iota \otimes \mathrm{id} : \mathcal{D}'(M) \otimes_{\mathbb{R}} \mathcal{C}^\infty(M) \to \mathcal{A}(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{C}^\infty(M)$ *is $\mathcal{C}^\infty(M)$-balanced, i.e.,* $\iota \otimes \mathrm{id}$ *from (4.4) is well-defined.*

(ii) ι *is $\mathcal{C}^\infty(M)$-linear.*

Proof. Let $v \in \mathcal{D}'(M)$ and $f, g \in \mathcal{C}^\infty(M)$.
(i)⇒(ii): $\iota(fv) \otimes 1 = \iota \otimes \mathrm{id}(fv \otimes 1) = \iota \otimes \mathrm{id}(v \otimes f) = \iota(v) \otimes f = f\iota(v) \otimes 1$. Thus, since $u \otimes f \mapsto fu$ is an isomorphism from $\mathcal{A}(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{C}^\infty(M)$ to $\mathcal{A}(M)$, (ii) follows.
(ii)⇒(i): $\iota \otimes \mathrm{id}(fv \otimes g) = \iota(fv) \otimes g = f\iota(v) \otimes g = \iota(v) \otimes fg = \iota \otimes \mathrm{id}(v \otimes fg)$. ☐

It is instructive to take a look at the coordinate version of the impossibility of (4.4). Indeed as we shall show below condition (i) of Proposition 4.1 is equivalent to the statement that coordinate-wise embedding of distributional tensor fields is independent of the choice of a local basis (cf. also [9]).

To this end, assume that M can be described by a single chart. Then $\mathcal{T}^r_s(M)$ has a $\mathcal{C}^\infty(M)$-basis consisting of (smooth) tensor fields, say, $e_1, \dots, e_m \in \mathcal{T}^r_s(M)$ with $m = n^{r+s}$. By (2.2), every $v \in \mathcal{D}'^r_s(M)$ can be written as $v = v^i \otimes e_i$ (using summation convention) with $v^i \in \mathcal{D}'(M)$. Consider a change of basis given by $e_i = a^j_i \hat{e}_j$, with a^j_i smooth. Then $v = \hat{v}^j \otimes \hat{e}_j$ with $\hat{v}^j = a^j_i v^i$. Applying $\iota \otimes \mathrm{id}$ to both representations of v, we obtain

$$(\iota \otimes \mathrm{id})(v^i \otimes e_i) = \iota(v^i) \otimes (a^j_i \hat{e}_j) = (\iota(v^i) a^j_i) \otimes \hat{e}_j = (\iota(a^j_i)\iota(v^i)) \otimes \hat{e}_j$$

resp.

$$(\iota \otimes \mathrm{id})(\hat{v}^j \otimes \hat{e}_j) = \iota(a_i^j v^i) \otimes \hat{e}_j$$

which are different in general due to (4.1). It follows that coordinate-wise embedding is not feasible for obtaining an embedding of tensor distributions.

The following example gives an explicit contradiction for the case $\mathcal{A}(M) = \hat{\mathcal{G}}(M)$.

Example 4.2. Set $M = \mathbb{R}$, and let $v \in \mathcal{D}'^1_0(\mathbb{R}) = \mathcal{D}'(\mathbb{R}) \otimes_{\mathcal{C}^\infty(\mathbb{R})} \mathfrak{X}(\mathbb{R})$ be given by $v = \delta' \otimes \partial_x$. Then

$$v = (1 + x^2)\delta' \otimes \frac{1}{1 + x^2}\partial_x$$

and we note that $(1 + x^2)$ is in fact the transition function of the underlying vector bundle TM with respect to the coordinate transformation $x \mapsto x + x^3$. With $\iota : \mathcal{D}'(\mathbb{R}) \to \hat{\mathcal{G}}(\mathbb{R})$, suppose that

$$\iota(\delta') \otimes \partial_x = \iota((1 + x^2)\delta') \otimes \frac{1}{1 + x^2}\partial_x.$$

Then since $x^2\delta' = 0$ in $\mathcal{D}'(\mathbb{R})$, this would amount to $(1 + x^2)\iota(\delta') = \iota(\delta')$. However, it is easily seen that x^2 is not a zero-divisor in $\hat{\mathcal{G}}(\mathbb{R})$ (adapt [18, Ex. 1.2.40] by choosing an appropriate smoothing kernel), so we arrive at a contradiction.

In order to circumvent the "domain obstruction" met in (4.4) (which arose from $\iota^r_s \otimes \mathrm{id}$ not being $\mathcal{C}^\infty(M)$-balanced) one might try to switch to isomorphic representations of the spaces involved: By (2.2) and (2.3), we have $\mathcal{D}'(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^r_s(M) \cong \mathrm{L}_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{D}'(M))$, and similarly $\mathcal{A}(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^r_s(M) \cong \mathrm{L}_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{A}(M))$ holds (the latter is proved analogously to the corresponding statement in [18, Th. 3.1.12]). The most plausible candidate for an embedding of $\mathrm{L}_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{D}'(M))$ into $\mathrm{L}_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{A}(M))$ certainly is ι_*, that is, composition from the left with $\iota : \mathcal{D}'(M) \to \mathcal{A}(M)$. Indeed, this choice presents no difficulties whatsoever with respect to the domain $\mathrm{L}_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{D}'(M))$. However, this time we encounter a "range obstruction" in the sense that we do end up only in $\mathrm{L}_{\mathbb{R}}(\mathcal{T}^s_r(M), \mathcal{A}(M))$, due to the fact that ι is only $\mathbb{R}$-linear. Proposition 4.1 demonstrates that the domain and the range obstructions, though of essentially different appearance, are in fact equivalent.

It is noteworthy that the range obstruction is encountered once more when trying to write down plausible formulae for an embedding of tensor distributions into a naïvely defined basic space for generalized tensor fields. Aiming at minimal changes as compared to the scalar theory it is natural to

start out from scalar basic space members $u : \hat{\mathcal{A}}_0(M)\times M \to \mathbb{R}$, to replace the "scalar" range space $\mathbb{R}$ by the vector bundle $\mathrm{T}^r_s M$ and to ask for $u(\omega, .)$ to be a member of $\mathcal{T}^r_s(M)$, for every $\omega \in \hat{\mathcal{A}}_0(M)$. Now when looking for a "tensor embedding" ι^r_s we aim at guaranteeing $\iota^r_s(v)(\omega, .)$ (for $v \in \mathcal{D}'^r_s(M)$) to be a member of $\mathcal{T}^r_s(M)$ by defining it via a $\mathcal{C}^\infty(M)$-linear action on $\tilde{t} \in \mathcal{T}^s_r(M)$. Virtually the only formula making sense is $\langle v, \tilde{t} \otimes \omega\rangle$, forcing us to set

$$(\iota^r_s(v)(\omega, .) \cdot \tilde{t})(p) := \langle v, \tilde{t} \otimes \omega\rangle. \tag{4.5}$$

At first glance, (4.5) displays a reassuring similarity to the scalar case definition $\iota(v)(\omega, p) := \langle v, \omega\rangle$. In particular, both right hand sides do not depend on p. This, however, leads to failure in the tensor case: Choosing $\tilde{t}$ with (nontrivial) compact support, the left hand side also has compact support with respect to p, so, being constant it has to vanish identically, making (4.5) absurd. On top of this and, in fact, continuing our above discussion we note that (4.5) also fails to provide $\mathcal{C}^\infty(M)$-linearity of $\iota^r_s(v)(\omega, .)$ since this would imply the contradictory relation ($f \in \mathcal{C}^\infty(M)$)

$$\langle v, (f\tilde{t}) \otimes \omega\rangle = (\iota^r_s(v)(\omega, .) \cdot (f\tilde{t}))(p) = f(p)\, (\iota^r_s(v)(\omega, .) \cdot \tilde{t})(p) = f(p)\, \langle v, \tilde{t} \otimes \omega\rangle.$$

Finally, (4.5) turns out to be nothing but a reformulation of the range obstruction: The element $\bar{v}$ of $\mathrm{L}_{\mathcal{C}^\infty(M)}(\mathcal{T}^r_s(M), \mathcal{D}'(M))$ corresponding to $v \in \mathcal{D}'^r_s(M)$ by $\langle \bar{v}(\tilde{t}), \omega\rangle = \langle v, \tilde{t} \otimes \omega\rangle$ satisfies $((\iota \circ \bar{v})(\tilde{t}))(\omega, p) = \langle \bar{v}(\tilde{t}), \omega\rangle = \langle v, \tilde{t} \otimes \omega\rangle$. Hence defining $\iota^r_s(v)$ by (4.5) corresponds to composing $\bar{v}$ with ι from the left which is the move leading straight into the range obstruction.

These considerations show that emulating the scalar case by naïve manipulation of formulae has to be abandoned. In the next section we show how the introduction of an additional geometric structure allows one to circumvent this problem. In particular, we will arrive at a formula for the embedding of tensor distributions ((5.3)) which allows a clear view on the failure of (4.5) and which, in fact, provides a remedy.

5 Previewing the construction

The obstructions to a component-wise embedding of distributional tensor fields discussed in the preceding section are essentially algebraic in nature. However, there is also a purely geometric reason for objecting to such an approach. We illustrate this below since it points the way toward the resolution of the problem, the basic idea going back to [42].

Let us begin by reviewing the embedding of a (regular) scalar distribution given by a continuous function g on M (see Definition 3.2). Pick some n-form

ω viewed as approximating the Dirac measure δ_p around $p \in M$. Then

$$(\iota g)(\omega, p) = \langle g, \omega \rangle = \int_M g(q)\omega(q)$$

may be seen as collecting values of g around p and forming a smooth average (recall that $\int \omega = 1$). Now, in case v is a continuous vector field, then its values $v(q)$ do not lie in the same tangent space for different q and there is in general no way of defining an embedding ι_0^1 of continuous vector fields via an integral of the form

$$\iota_0^1(v)(\omega, p) = \int_M v(q)\omega(q) \tag{5.1}$$

since there is no way of identifying $\mathrm{T}_p M$ and $\mathrm{T}_q M$ for $p \neq q$.

However, this observation also points the way to the remedy: we need some additional geometric structure providing such an identification. One possibility would be to use a (background) connection or Riemannian metric. Let p, q lie within a geodesically convex neighborhood. Then parallel transport along the unique geodesic connecting p and q defines a map $A(p, q) : \mathrm{T}_p M \to \mathrm{T}_q M$. In principle it would be possible to employ the shrinking supports of the smoothing kernels to extend this locally defined "transport operator" to the whole manifold using suitable cut-off functions. However, to avoid technicalities we have chosen to work directly with compactly supported transport operators A defined as compactly supported smooth sections of the bundle $\mathrm{TO}(M, M) = \mathrm{L}_{M\times M}(TM, TM)$ (see Appendix A), i.e., $A(p, q)$ being a linear map $\mathrm{T}_p M \to \mathrm{T}_q M$. This map may be used to "gather" at p the values of v (via $A(q, p)v(q)$) before averaging them, i.e., we may set

$$\iota_0^1(v)(\omega, p, A) := \int_M A(q, p)v(q)\,\omega(q), \tag{5.2}$$

with the new mechanism becoming most visible by comparing (5.1) with (5.2).

Observe, however, the following important fact: To maintain the spirit of the full construction, i.e., to provide a canonical embedding independent of additional choices we have to make the elements of our basic space depend on an additional third slot containing A. Indeed, as one can show, $\iota_0^1(v)$ as defined in (5.2) above depends smoothly on ω, p, A. (In fact, the proof of this statement in the general case is one of the technically most demanding parts of this paper and will be given in Section 7.) Thus for each fixed pair (ω, A) we have that

$$\iota_0^1(v)(\omega, A) := [p \mapsto \iota_0^1(v)(\omega, p, A)]$$

defines a smooth vector field on M. This strongly suggests that we choose our basic space $\hat{\mathcal{E}}^1_0(M)$ of generalized vector fields to explicitly include dependence on the transport operators, i.e.,

$$\hat{\mathcal{E}}^1_0(M) := \{u \in \mathcal{C}^\infty(\hat{\mathcal{A}}_0(M) \times M \times \Gamma_c(\mathrm{TO}(M,M)), \mathrm{T}M) \mid u(\omega,p,A) \in \mathrm{T}_pM\}.$$

In particular, $p \mapsto t(\omega,p,A)$ is a member of $\mathfrak{X}(M)$ for any fixed ω, A. Following this strategy of course means that one also has to allow for dependence of scalar fields on transport operators and one must therefore upgrade the scalar theory from the old 2-slot version as presented in Section 3 to a new 3-slot version.

Finally, we may turn to embedding general distributional vector fields. By definition of $\mathcal{D}'^1_0(M)$, v takes (finite sums of) tensors $\tilde{u} \otimes \omega$ with $\tilde{u} \in \Omega^1(M)$ as arguments. Hence the most convenient way of defining $\iota^1_0(v)(\omega,p,A)$ is to let the prospective smooth vector field $\iota^1_0(v)(\omega,A)$ act on a one-form $\tilde{u}$. In fact, we may write for continuous v

$$\begin{aligned}
\iota^1_0(v)(\omega,p,A)\cdot\tilde{u}(p) &= \big(\iota^1_0(v)(\omega,A)\cdot\tilde{u}\big)(p)\\
&= \int_M A(q,p)v(q)\cdot\tilde{u}(p)\ \omega(q)\\
&= \int_M v(q)\cdot A(q,p)^{\mathrm{ad}}\tilde{u}(p)\ \omega(q)\\
&= \langle v(\,.\,), A(\,.\,,p)^{\mathrm{ad}}\tilde{u}(p)\otimes\omega(\,.\,)\rangle.
\end{aligned}$$

In the last expression above, we are now free to replace the regular distributional vector field v by any $v \in \mathcal{D}'^1_0(M)$. This leads to our definition of ι^1_0 by

$$\begin{aligned}
\iota^1_0(v)(\omega,p,A)\cdot\tilde{u}(p) &:= \big(\iota^1_0(v)(\omega,A)\cdot\tilde{u}\big)(p)\\
&:= \langle\ v(.)\ ,A(\,.\,,p)^{\mathrm{ad}}\tilde{u}(p)\ \otimes\ \omega(.)\ \rangle.
\end{aligned} \tag{5.3}$$

Observe the shift of focus in the above formulas as compared to (5.2): rather than thinking of the transport operator as "gathering" at p the values of the vector field v it (more precisely, its flipped and adjoint version) serves to "spread" the value of the "test one-form" $\tilde{u}(p)$ at p to the neighboring points q.

Connecting to Section 4 we point out that the embedding (5.3) may be viewed as a correction of the flawed formula (4.5). Comparison reveals that the introduction of the transport operator, i.e., the replacement of $\tilde{u}(.)$ by $A(\,.\,,p)^{\mathrm{ad}}\tilde{u}(p)$, removes both failures of (4.5): the right hand side now does depend on p and, moreover, defines a $\mathcal{C}^\infty(M)$-linear mapping on $\Omega^1(M)$.

The case of general (r,s)-tensor fields can be dealt with by using appropriate tensor products of the transport operators. The details of this are

given in Section 6 on the kinematics part of our construction. In particular, this includes the definition of a basic space for generalized tensor fields of type (r, s) which depend on transport operators and the general definition of the embeddings σ^r_s of smooth and ι^r_s of distributional tensor fields. Furthermore we define the pullback action as well as the Lie derivative with respect to smooth vector fields for elements of the basic space in such a way that they commute with the embeddings. An added complication as compared to the scalar case is the fact that the transport operators are two-point objects so that the action of diffeomorphisms needs to be treated with some care. Some basic material on this topic is collected in Appendix A.

As already indicated above the proof that the embedded image $\iota^r_s(v)$ of a distributional tensor field v is smooth with respect to all its three variables (hence belongs to the basic space) is rather involved. It builds on some results on calculus in (infinite-dimensional) convenient vector spaces which are nontrivial to derive for the following reason: We have to carefully distinguish (and bridge the gap) between the standard locally convex topologies defined on the respective spaces of sections and their convenient structures on which the calculus according to [26] rests. We provide the proof of smoothness of $\iota^r_s(v)$ in Section 7 and have deferred some useful results on the calculus to Appendix B.

The dynamics part of our construction is carried out in Section 8. The heart of this part is the quotient construction that allows one to identify $\iota^r_s(v)$ and $\sigma^r_s(v)$ for smooth tensor fields v. The introduction of the transport operator as a variable means that the "scalar" space $\hat{\mathcal{E}}^0_0(M)$ has to be refined as compared to $\hat{\mathcal{E}}(M)$ from [19] by introducing a third argument. However we can connect the present scalar theory to that in [19] by using an appropriate saturation principle (Proposition 8.8). Since generalized tensor fields depend on transport operators, derivatives with respect to these have to be taken into account as well. Fortunately, due to a reduction principle (Lemma 8.6) these derivatives decouple from the others. This fact allows to directly utilize results from [19] without having to rework the local theory from [17] in the present context.

An important feature of the Colombeau algebras in the scalar case is an equivalence relation known as "association" which coarse grains the algebra. As we remarked earlier the Schwartz impossibility result means that one cannot expect that for general *continuous* functions the pointwise product commutes with the embedding. However this result is true at the level of association. Furthermore in many situations of practical relevance elements of the algebra are associated to conventional distributions. In applications, this feature has the advantage that in many cases one may use the mathematical power of the differential algebra to perform calculations but then use

the notion of association to give a physical interpretation to the answer. In Section 9 we extend the definition of association from the scalar to the tensor case and show in particular that the tensor product of continuous tensor fields commutes with the embedding at the level of association.

6 Kinematics

In this section we introduce the basic space for the forthcoming spaces of generalized sections. We also define the embeddings of smooth and distributional sections as well as the action of diffeomorphisms and the Lie derivative. The main result of this section is that the Lie derivative commutes with the embedding of distributions already at the level of the basic space.

We begin by collecting the ingredients for the definition of the basic space. For the space $\hat{\mathcal{A}}_0(M)$ we refer to Definition 3.1 (i), and for details on the space of transport operators $\Gamma(\mathrm{TO}(M,N))$ to Appendix A.

Definition 6.1. We define the space of compactly supported transport operators on M by

$$\hat{\mathcal{B}}(M) := \Gamma_{\mathrm{c}}(\mathrm{TO}(M,M)).$$

Elements of $\hat{\mathcal{A}}_0(M)$ resp. $\hat{\mathcal{B}}(M)$ will generically be denoted by ω resp. A.

Definition 6.2. The basic space for generalized sections of type (r,s) on the manifold M is defined as

$$\hat{\mathcal{E}}^r_s(M) := \{u \in \mathcal{C}^\infty(\hat{\mathcal{A}}_0(M) \times M \times \hat{\mathcal{B}}(M), \mathrm{T}^r_s M) \mid u(\omega,p,A) \in (\mathrm{T}^r_s)_p M\}.$$

Here, both $\hat{\mathcal{A}}_0(M)$ and $\hat{\mathcal{B}}(M)$ are equipped with their natural (LF)-topologies in the sense of Section 2. Recall that smoothness is to be understood in the sense of [26]. In particular, $u(\omega,A) := p \mapsto u(\omega,p,A)$ is a member of $\mathcal{T}^r_s(M)$ for ω, A fixed.

We remark that the definition aimed at in [42] used two-point tensors ("TP", see Appendix A) rather than transport operators ("TO"). Of course, it is always possible to switch from the "TO-picture" to the "TP-picture" by means of the isomorphism given in (A.2).

Next we introduce a core technical device for embedding distributional sections of $\mathrm{T}^r_s M$ into the basic space.

Definition 6.3. Given $A \in \hat{\mathcal{B}}(M)$ we denote by $A^s_r(p,q)$ the induced linear map from $(\mathrm{T}^s_r)_p M$ to $(\mathrm{T}^s_r)_q M$, i.e., for any $\tilde{t}_p = w_1 \otimes \cdots \otimes w_s \otimes \beta^1 \otimes \cdots \otimes \beta^r \in (\mathrm{T}^s_r)_p M$ we write

$$A^s_r(p,q)(\tilde{t}_p) := A(p,q)w_1 \otimes \cdots \otimes A(q,p)^{\mathrm{ad}}\beta^r \in (\mathrm{T}^s_r)_q M. \tag{6.1}$$

Obviously, for all $\tilde{t} \in \mathcal{T}_r^s(M)$, the map $q \mapsto A_r^s(p,q)\,\tilde{t}(p) := A_r^s(p,q)(\tilde{t}(p))$ again defines an element of $\mathcal{T}_r^s(M)$, for every fixed $p \in M$. Moreover, given a second manifold N, it should be clear how to generalize the definition of A_r^s to the case of $A \in \Gamma(\mathrm{TO}(M,N))$. Assigning to $(\tilde{t}_p, A) \in (\mathrm{T}_r^s)_p M \times \Gamma(\mathrm{TO}(M,N))$ the (smooth) tensor field $(q \mapsto A_r^s(p,q)\,\tilde{t}_p) \in \mathcal{T}_r^s(N)$ will be referred to as "spreading $\tilde{t}_p$ over N via A". Dually, assigning to $(t,p,A) \in \mathcal{T}_s^r(N) \times M \times \Gamma(\mathrm{TO}(M,N))$ the map $q \mapsto A_r^s(p,q)^{\mathrm{ad}}\,t(q) \in (\mathrm{T}_s^r)_p M$ (being defined on N) will be referred to as "gathering t at p via A" (compare also Section 5).

Definition 6.4.

(i) We define the embedding $\sigma_s^r : \mathcal{T}_s^r(M) \to \hat{\mathcal{E}}_s^r(M)$ of smooth sections of $\mathrm{T}_s^r M$ into the basic space $\hat{\mathcal{E}}_s^r(M)$ by

$$\sigma_s^r(t)(\omega, A) := t$$

resp.

$$\sigma_s^r(t)(\omega, p, A) := t(p).$$

(ii) We define the embedding $\iota_s^r : \mathcal{D}'^r_s(M) \to \hat{\mathcal{E}}_s^r(M)$ of distributional sections of $\mathrm{T}_s^r M$ into the basic space $\hat{\mathcal{E}}_s^r(M)$ via its action on sections $\tilde{t} \in \mathcal{T}_r^s(M)$ by

$$(\iota_s^r(v)(\omega, A) \cdot \tilde{t})(p) = \iota_s^r(v)(\omega, p, A) \cdot \tilde{t}(p) := \langle v(.), \big(A_r^s(p,.)\,\tilde{t}(p)\big) \otimes \omega(.)\rangle.$$

In contrast to the case of $\sigma_s^r(t)$ where $p \in M$ can simply be plugged into $t \in \mathcal{T}_s^r(M)$, the variable p is not a natural ingredient of the argument of a distribution $v \in \mathcal{D}'^r_s(M)$. Consequently, it only occurs as a parameter in the definition of $\iota_s^r(v)$. Therefore, a p-free version of the definition of $\iota_s^r(v)$ giving meaning directly to $\iota_s^r(v)(\omega, A)$ is not feasible. On the other hand, the occurrence of $\tilde{t} \in \mathcal{T}_r^s(M)$ in the definition of $\iota_s^r(v)$ is essentially due to the fact that v requires tensors $\tilde{t} \otimes \omega$ with $\tilde{t} \in \mathcal{T}_r^s(M)$ and $\omega \in \Omega_c^n(M)$ to be fed in as arguments. A $\tilde{t}$-free version of the definition of ι_s^r, however, is in fact feasible, cf. Remark 7.5 below.

It is clear that σ_s^r is linear, taking elements of $\hat{\mathcal{E}}_s^r(M)$ as values. As to ι_s^r, the map A_r^s given by equation (6.1) together with $\tilde{t} \in \mathcal{T}_r^s(M)$ produce a smooth section $A_r^s(p,.)\,\tilde{t}(p)$ of $\mathrm{T}_r^s M$, with p as parameter. Hence the action of v on $A_r^s(p,.)\,\tilde{t}(p) \otimes \omega(.)$ is defined, giving a complex number depending on p. Since $\iota_s^r(v)(\omega, p, A)$ is linear in $\tilde{t}(p)$ and $\tilde{t}(p)$ was arbitrary, $\iota_s^r(v)(\omega, p, A) \in (\mathrm{T}_s^r)_p M$. To prove the fact that $\iota_s^r(v)$ is a smooth function of its three arguments (in the sense of [26]), hence in fact takes values in $\hat{\mathcal{E}}_s^r(M)$

is more delicate and will be postponed until Section 7. Moreover, equipping ${\mathcal{D}'}^r_s(M)$ and $\hat{\mathcal{E}}^r_s(M)$ with the respective topologies of pointwise convergence (on $\mathcal{T}_r^s(M) \otimes_{\mathcal{C}^\infty(M)} \Omega^n_c(M)$ resp. on $\hat{\mathcal{A}}_0(M) \times M \times \hat{\mathcal{B}}(M)$), the embedding ι^r_s is linear and bounded, hence smooth by [26, 2.11]. By the uniform boundedness principle stated in [26, 30.3], ι^r_s remains smooth when the range space is equipped with the (C)-topology as defined in Appendix B. By similar (in fact, easier) arguments, σ^r_s is smooth in the same sense. Finally, injectivity of ι^r_s is a consequence of Theorem 8.12 (iv) below. A direct proof, not involving the tools of Section 8, is possible, yet for the sake of brevity we refrain from including it.

Next we turn to the action of diffeomorphisms on the basic space and the diffeomorphism invariance of the embedding ι^r_s. To begin with we take a look at the transformation behavior of the map $A^s_r(p,q)$ under diffeomorphisms. In fact, as it turns out in the context of Lie derivatives (cf. the proof of the key Proposition 6.8 below) it is necessary to use a concept allowing for the simultaneous action of two different diffeomorphisms at either slot of A. This corresponds to the natural action of pairs of diffeomorphisms[1] on transport operators as defined in (A.3).

So let $\mu, \nu : M \to N$ be diffeomorphisms. By equation (A.3) we have the following induced action on the factors of $A^s_r(p,q)$:

$$\begin{aligned} \big((\mu,\nu)^* A\big)(p,q) &= (T_q\nu)^{-1} \circ A\big(\mu(p),\nu(q)\big) \circ T_p\mu && (6.2)\\ \big((\mu,\nu)^* A\big)(q,p)^{\mathrm{ad}} &= (T_q\mu)^{\mathrm{ad}} \circ A\big(\mu(q),\nu(p)\big)^{\mathrm{ad}} \circ (T_p\nu)^{-1,\mathrm{ad}}, && (6.3)\end{aligned}$$

and the action on A^s_r is given by

$$(\mu,\nu)^*\big(A^s_r\big)(p,q) = \big((\mu,\nu)^* A\big)^s_r(p,q). \qquad (6.4)$$

Definition 6.5. Let $\mu : M \to N$ be a diffeomorphism. We define the induced action of μ on the basic space, $\hat\mu^* : \hat{\mathcal{E}}^r_s(N) \to \hat{\mathcal{E}}^r_s(M)$, by

$$\begin{aligned} \big(\hat\mu^* u\big)(\omega,p,A) &:= \mu^*\Big(u\big(\mu_*\omega,(\mu,\mu)_*A\big)\Big)(p)\\ &= \big(T_{\mu(p)}\mu^{-1}\big)^r_s\, u\big(\mu_*\omega, \mu p, (\mu,\mu)_*A\big).\end{aligned}$$

It is clear that $\hat\mu^* u$ assigns a member of $(\mathrm{T}^r_s)_p M$ to every (ω,p,A). In order to obtain $\hat\mu^* u \in \hat{\mathcal{E}}^r_s(M)$, we have to establish smoothness in (ω,p,A). Observing support properties and (2.1) it follows that the linear maps $\omega \mapsto \mu_*\omega$ and $A \mapsto (\mu,\mu)_*A$ are bounded (equivalently, smooth, by [26, 2.11]) with

[1] yet not of arbitrary diffeomorphisms of $\rho : M_1 \times N_1 \to M_2 \times N_2$, cf. the discussion following (A.5) and (A.8)

respect to the (LF)-topologies. Since u and the action of $\mathrm{T}\mu^{-1}$ on $\mathrm{T}^r_s M$ are also smooth, we see that indeed $\hat{\mu}^* u \in \hat{\mathcal{E}}^r_s(M)$ holds.

To facilitate the proof of the next proposition we introduce the following notation: For $A \in \Gamma(\mathrm{TO}(M,N))$, $\tilde{t} \in \mathcal{T}^s_r(M)$, $p \in M$ denote the spreading $q \mapsto A^s_r(p,q)\,\tilde{t}(p)$ of $\tilde{t}(p)$ via A by $\theta(A,\tilde{t},p) \in \mathcal{T}^s_r(N)$. It is easy to check that for $\mu : M \to N$, $\tilde{t} \in \mathcal{T}^s_r(N)$ and $A \in \Gamma(\mathrm{TO}(N,N))$,

$$\big((\mu,\mu)^* A^s_r\big)(p,q) \cdot (\mu^* \tilde{t})(p) = \mu^*\big(\theta(A,\tilde{t},\mu p)\big)(q).$$

Moreover, using θ we may write for $A \in \hat{\mathcal{B}}(M)$

$$\big(\iota^r_s(v)(\omega,A) \cdot \tilde{t}\big)(p) = \langle v, \theta(A,\tilde{t},p) \otimes \omega\rangle \qquad (v \in \mathcal{D}'^r_s(M)).$$

Proposition 6.6. *The action of diffeomorphisms commutes with the embedding ι^r_s, that is, we have for all $v \in \mathcal{D}'^r_s(N)$ and all diffeomorphisms $\mu : M \to N$*

$$\hat{\mu}^* \iota^r_s(v) = \iota^r_s(\mu^* v). \tag{6.5}$$

Proof. Let $\mu : M \to N$ be a diffeomorphism and let $v \in \mathcal{D}'^r_s(N)$, $\omega \in \hat{\mathcal{A}}_0(M)$, $A \in \hat{\mathcal{B}}(M)$, $\tilde{t} \in \mathcal{T}^s_r(M)$, and $p \in M$. Then we have

$$\begin{aligned}
&\Big(\big(\hat{\mu}^* \iota^r_s(v)\big)\big(\omega, A\big) \cdot \tilde{t}\Big)(p) \\
&\qquad = \Big(\mu^*\big(\iota^r_s(v)(\mu_*\omega, (\mu,\mu)_* A)\big) \cdot \tilde{t}\Big)(p) \\
&\qquad = \mu^*\Big(\iota^r_s(v)\big(\mu_*\omega, (\mu,\mu)_* A\big) \cdot \mu_* \tilde{t}\Big)(p) \\
&\qquad = \Big(\iota^r_s(v)(\mu_*\omega, (\mu,\mu)_* A) \cdot \mu_* \tilde{t}\Big)(\mu p) \\
&\qquad = \langle v(.), \big(((\mu,\mu)_* A^s_r)(\mu p, .)\, (\mu_* \tilde{t})(\mu p)\big) \otimes \mu_*\omega(.)\rangle \\
&\qquad = \langle v(.), \mu_*\big(\theta(A,\tilde{t},p)\big)(.) \otimes \mu_*\omega(.)\rangle \\
&\qquad = \langle (\mu^* v)(..), \theta(A,\tilde{t},p)(..) \otimes \omega(..)\rangle \\
&= \Big(\big(\iota^r_s(\mu^* v)(\omega, A)\big) \cdot \tilde{t}\Big)(p).
\end{aligned}$$

$\square$

Next we turn to the Lie derivative on the basic space $\hat{\mathcal{E}}^r_s(M)$. To begin with suppose that X is a smooth and complete vector field on M so that the flow Fl^X is defined globally on $\mathbb{R} \times M$. Then we may use Definition 6.5 to define the Lie derivative of $u \in \hat{\mathcal{E}}^r_s(M)$ via

$$\hat{\mathrm{L}}_X u := \frac{\mathrm{d}}{\mathrm{d}\tau}\Big|_{\tau=0} (\widehat{\mathrm{Fl}^X_\tau})^* u. \tag{6.6}$$

In the sequel, we will write $\widehat{\mathrm{Fl}}_\tau^X$ instead of (the correct) $\widehat{\mathrm{Fl}_\tau^X}$, for the sake of line spacing. For $(\hat{\mathrm{L}}_X u)(\omega, p, A)$ to exist (as an element of $(\mathrm{T}^r_s)_p M$) it suffices to know that $\tau \mapsto (\widehat{\mathrm{Fl}}_\tau^X)^* u(\omega, p, A)$ is smooth. However, for $\hat{\mathrm{L}}_X u$ to exist and to be a member of $\hat{\mathcal{E}}^r_s(M)$ (i.e., to be a smooth function of its arguments (ω, p, A)) we even need that $(\tau, \omega, p, A) \mapsto ((\widehat{\mathrm{Fl}}_\tau^X)^* u)(\omega, p, A) = T^r_s \mathrm{Fl}^X_{-\tau}(u((\mathrm{Fl}^X_\tau)_*\omega, \mathrm{Fl}^X_\tau p, (\mathrm{Fl}^X_\tau)_* A))$ is smooth on $(-\tau_0, +\tau_0) \times \hat{\mathcal{A}}_0(M) \times M \times \hat{\mathcal{B}}(M)$ for some $\tau_0 > 0$. Indeed, by Proposition A.2 (1), $(\tau, \omega) \mapsto (\mathrm{Fl}^X_\tau)_*\omega$ and $(\tau, A) \mapsto (\mathrm{Fl}^X_\tau)_* A$ are smooth, as is $(\tau, p) \mapsto \mathrm{Fl}^X_\tau p$. Moreover, a local argument shows that the action of Fl^X on $\mathbb{R} \times \mathrm{T}^r_s M$ sending (τ, v) to $T^r_s \mathrm{Fl}^X_{-\tau} v$ is smooth (this in fact also secures that the assumptions of Proposition A.2 (1) are satisfied). Together with smoothness of u, we indeed obtain $\hat{\mathrm{L}}_X u \in \hat{\mathcal{E}}^r_s(M)$.

Note that in order to have $(\widehat{\mathrm{Fl}}_\tau^X)^* u$ defined as a member of $\hat{\mathcal{E}}^r_s(M)$ even for only one particular value of τ we need $(\mathrm{Fl}^X_\tau)_*\omega$, $\mathrm{Fl}^X_\tau p$ and $(\mathrm{Fl}^X_\tau)_* A)$ defined for this τ and for *all* ω, p, A, irrespective of the size of the supports of ω and A (as to p, we could content ourselves with some open subset of M). This exhibits the important role of completeness of X in the geometric approach to the Lie derivative on the basic space taken by (6.6).

As a direct consequence of (6.6) and Proposition 6.6 we have that the Lie derivative commutes with the embedding, i.e., we have for all smooth and complete vector fields X and all $v \in \mathcal{D}'^r_s(M)$

$$\hat{\mathrm{L}}_X \iota^r_s(v) = \iota^r_s(\mathrm{L}_X v). \tag{6.7}$$

The technical background of passing from (6.5) to (6.7) again involves calculus in convenient vector spaces. For $v \in \mathcal{D}'^r_s(M)$, $\tau \mapsto (\mathrm{Fl}^X_\tau)^* v$ is smooth and $\mathrm{L}_X v = \frac{\mathrm{d}}{\mathrm{d}\tau}\big|_0 (\mathrm{Fl}^X_\tau)^* v$ holds with respect to the weak topology, due to (1)(ii) of Proposition A.2. Also, $\hat{\mathrm{L}}_X u = \frac{\mathrm{d}}{\mathrm{d}\tau}\big|_0 (\widehat{\mathrm{Fl}}^X_\tau)^* u$ for $u = \iota^r_s(v)$ with respect to the topology of pointwise convergence. Applying the chain rule [26, 3.18] to the function $\tau \mapsto (\iota^r_s \circ (\mathrm{Fl}^X_\tau)^*)v$, we obtain from (6.5):

$$\begin{aligned}\hat{\mathrm{L}}_X \iota^r_s(v) &= \tfrac{\mathrm{d}}{\mathrm{d}\tau}\big|_0 ((\widehat{\mathrm{Fl}}^X_\tau)^*(\iota^r_s(v))) = \tfrac{\mathrm{d}}{\mathrm{d}\tau}\big|_0 (\iota^r_s((\mathrm{Fl}^X_\tau)^* v)) = \iota^r_s(\tfrac{\mathrm{d}}{\mathrm{d}\tau}\big|_0 (\mathrm{Fl}^X_\tau)^* v) \\ &= \iota^r_s(\mathrm{L}_X v).\end{aligned}$$

For the purpose of extending the definition of the Lie derivative to arbitrary smooth vector fields, by an application of the chain rule we obtain from (6.6)

$$\begin{aligned}&(\hat{\mathrm{L}}_X u)(\omega, p, A) \\ &\quad = \mathrm{L}_X(u(\omega, A))(p) - \mathrm{d}_1 u(\omega, p, A)(\mathrm{L}_X \omega) - \mathrm{d}_3 u(\omega, p, A)(\mathrm{L}_X A),\end{aligned} \tag{6.8}$$

where we recall from Appendix A that $\mathrm{L}_X A$ is an abbreviation for $\mathrm{L}_{X,X} A$.

Note that we do not need full manifold versions of local results of infinite-dimensional calculus as, e.g., the chain rule [26, 3.18] since we can replace $\hat{\mathcal{A}}_0(M) \times M \times \hat{\mathcal{B}}(M)$ by $\hat{\mathcal{A}}_{00}(M) \times W \times \hat{\mathcal{B}}(M)$ when dealing with local issues on M. Here, $\hat{\mathcal{A}}_{00}(M)$ denotes the linear subspace of $\Omega^n_{\mathrm{c}}(M)$ parallel to $\hat{\mathcal{A}}_0(M)$ and W is (diffeomorphic to) some open subset of $\mathbb{R}^n$. In this way, $\mathrm{d}_1 u(\omega, p, A)(\eta)$, for example, can be interpreted locally as $\mathrm{d}u(\omega, p, A)(\eta, 0, 0)$ in the above sense or, equivalently, as $\mathrm{d}(u^\vee(p, A))(\omega)(\eta)$ where $u^\vee(p, A)(\omega) = u(\omega, p, A)$.

The scalar analogue of (6.8) first appeared in the local setting of [21, Rem. 22], where it arises as an operational consequence of Jelínek's approach; see also the discussion of the scalar case in [38, p. 4]. Here, however, it is a direct consequence of our natural choice of definitions, in case X is complete. In the general case we turn equation (6.8) into a definition.

Definition 6.7. Given a smooth vector field X on M we define the Lie derivative $\hat{\mathrm{L}}_X$ with respect to X on the basic space by

$$\begin{aligned} (\hat{\mathrm{L}}_X u)(\omega, p, A) \;&=\; \mathrm{L}_X(u(\omega, A))(p) \\ &\quad -\mathrm{d}_1 u(\omega, p, A)(\mathrm{L}_X \omega) - \mathrm{d}_3 u(\omega, p, A)(\mathrm{L}_X A). \end{aligned} \tag{6.9}$$

To see that the first term on the right hand side of (6.9) actually defines an element of $\hat{\mathcal{E}}^r_s(M)$ we note that by Corollary B.11 (resp. Lemma 7.2 and Remark B.3), $\hat{\mathcal{E}}^r_s(M)$ is linearly isomorphic to $\mathcal{C}^\infty(\hat{\mathcal{A}}_0(M) \times \hat{\mathcal{B}}(M), \mathcal{T}^r_s(M))$ where $\mathcal{T}^r_s(M)$ carries the (F)-topology. Since L_X is linear and bounded (hence smooth) on $\mathcal{T}^r_s(M)$, the map $(\omega, A) \mapsto \mathrm{L}_X(u(\omega, A))$ is smooth from $\hat{\mathcal{A}}_0(M) \times \hat{\mathcal{B}}(M)$ into $\mathcal{T}^r_s(M)$. By smoothness of $\omega \mapsto \mathrm{L}_X \omega$ resp. $A \mapsto \mathrm{L}_X A$, also the second and the third term define members of the basic space. Note that in the scalar case $r = s = 0$, the first term takes the form of a directional derivative as well, to wit, $\mathrm{d}_2 u(\omega, p, A)(X)$.

For complete $X \in \mathfrak{X}(M)$, also the three individual terms at the right hand side of (6.9) arise in a geometric way: By slightly generalizing the constructions of $\widehat{\mathrm{Fl}}^X_\tau$ and $\hat{\mathrm{L}}_X$ for complete X we can define, for given complete vector fields $X, Y, Z \in \mathfrak{X}(M)$, also $(\widehat{\mathrm{Fl}}^{X,Y,Z}_\tau)^* u := T^r_s \mathrm{Fl}^Y_{-\tau}(u((\mathrm{Fl}^X_\tau)_* \omega, \mathrm{Fl}^Y_\tau p, (\mathrm{Fl}^Z_\tau)_* A))$ and $\hat{\mathrm{L}}_{X,Y,Z}\, u := \frac{\mathrm{d}}{\mathrm{d}\tau}\big|_0 (\widehat{\mathrm{Fl}}^{X,Y,Z}_\tau)^* u$. From this we obtain, by the chain rule,

$$\hat{\mathrm{L}}_X u = \hat{\mathrm{L}}_{X,0,0} u + \hat{\mathrm{L}}_{0,X,0} u + \hat{\mathrm{L}}_{0,0,X} u. \tag{6.10}$$

All the smoothness arguments referring to $\hat{\mathrm{L}}_X$ equally apply to each individual term of this decomposition. Moreover, it is clear that the three terms occurring on the right hand side of (6.9) correspond to $\hat{\mathrm{L}}_{0,X,0}$, $\hat{\mathrm{L}}_{X,0,0}$, $\hat{\mathrm{L}}_{0,0,X}$,

respectively. Therefore we will retain this notation also in the case of arbitrary vector fields. It is immediate that $(\hat{\mathrm{L}}_{0,X,0}u)(\omega,p,A) = \mathrm{L}_X(u(\omega,A))(p)$ where L_X denotes the classical Lie derivative on $\mathcal{T}^r_s(M)$. On the other hand, in $\hat{\mathrm{L}}_{X,0,0}u$ and $\hat{\mathrm{L}}_{0,0,X}u$ the p-slot is fixed and the differentiation process involves only the fiber $(\mathrm{T}^r_s)_pM \cong (\mathbb{R}^n)^{r+s}$ as range space. At several places it will be important to split $\hat{\mathrm{L}}_X$ in the way just indicated (to wit, in 6.8, 6.9, 8.13, 8.14).

In the case of an arbitrary smooth vector field X some work is needed to prove that the embedding commutes with the Lie derivative. In Appendix A (see (A.6) and the remark following (A.8)) we have defined the Lie derivative of $A \in \Gamma(\mathrm{TO}(M,N))$ with respect to any smooth vector fields X, Y. We now set

$$\begin{aligned} \mathrm{L}_{X,Y}A^s_r &= \mathrm{L}_{X,Y}A \otimes \cdots \otimes (A^{\mathrm{ad}} \circ \mathrm{fl}) + \ldots \\ &\qquad\qquad \cdots + A \otimes \cdots \otimes \mathrm{L}_{X,Y}(A^{\mathrm{ad}} \circ \mathrm{fl}), \end{aligned}$$

where fl denotes the flip $(p,q) \mapsto (q,p)$. This conforms to viewing A^s_r as a section of the vector bundle over $M \times N$ with fiber $L((\mathrm{T}^s_r)_pM, (\mathrm{T}^s_r)_qN)$ at (p,q) and the obvious transition functions. We will use the notation $\mathrm{L}_{X,Y}(A^s_r(p,q)\,\tilde{t}(p))$ rather than (the more precise) $\mathrm{L}_{X,Y}(A^s_r(p,q)(\mathrm{pr}_1^*\tilde{t}(p,q)))$ for the Lie derivative of the section $(p,q) \mapsto A^s_r(p,q)\,\tilde{t}(p)$ of the pullback bundle.

Proposition 6.8. *The Lie derivative commutes with the embedding ι^r_s, i.e., we have, for all smooth vector fields X and all $v \in \mathcal{D}'^r_s(M)$,*

$$\hat{\mathrm{L}}_X \iota^r_s(v) = \iota^r_s(\mathrm{L}_X v). \tag{6.11}$$

Proof. By definition we have for all $\omega \in \hat{\mathcal{A}}_0(M)$, $p \in M$ and $A \in \hat{\mathcal{B}}(M)$

$$\begin{aligned} \big(\hat{\mathrm{L}}_X \iota^r_s(v)\big)(\omega,p,A) = \mathrm{L}_X\big(\iota^r_s(v)(\omega,A)\big)(p) \\ - \mathrm{d}_1\big(\iota^r_s(v)\big)(\omega,p,A)(\mathrm{L}_X\omega) \\ - \mathrm{d}_3\big(\iota^r_s(v)\big)(\omega,p,A)(\mathrm{L}_X A). \end{aligned} \tag{6.12}$$

We proceed by applying each term individually to $\tilde{t} \in \mathcal{T}^s_r(M)$. We find by the chain rule

$$\begin{aligned} &\mathrm{L}_X\big(\iota^r_s(v)(\omega,A)\big)(p)\cdot\tilde{t}(p) \\ &\quad = \mathrm{L}_X\big(\iota^r_s(v)(\omega,A)\cdot\tilde{t}\big)(p) - \big(\iota^r_s(v)(\omega,A)\cdot\mathrm{L}_X\tilde{t}\big)(p) \\ &\quad = \mathrm{L}_X\Big(\langle v(.), A^s_r(p,.)\,\tilde{t}(p)\otimes\omega(.)\rangle\Big) - \langle v(.), A^s_r(p,.)\,\mathrm{L}_X\tilde{t}(p)\otimes\omega(.)\rangle \\ &\quad = \langle v(.), \Big(\mathrm{L}_{X,0}\big(A^s_r(p,.)\,\tilde{t}(p)\big) - A^s_r(p,.)\,\mathrm{L}_X\tilde{t}(p)\Big)\otimes\omega(.)\rangle \\ &\quad = \langle v(.), (\mathrm{L}_{X,0}A^s_r)(p,.)\,\tilde{t}(p)\otimes\omega(.)\rangle. \end{aligned} \tag{6.13}$$

To see that $\mathrm{L}_X \circ v = v \circ \mathrm{L}_{X,0}$ in the above calculation, set $w(p,q) := A^s_r(p,q)\,\tilde{t}(p)$. Then we have $w \in \Gamma_c(M\times M, \mathrm{pr}_2^*\mathrm{T}^s_r M)$, with $w^\vee \in \mathcal{C}^\infty(M, \mathcal{T}_r^s(M))$ corresponding to w according to Lemma B.9. On $\Gamma_c(M \times M, \mathrm{pr}_2^*\mathrm{T}^s_r M)$ flow actions $(\mathrm{Fl}^X_\tau, \mathrm{Fl}^Y_\tau)^*$ and Lie derivatives $\mathrm{L}_{X,Y}$ are defined in complete analogy to the case of transport operators (Appendix A). Since $\operatorname{supp} w \subseteq \operatorname{supp} A$ there exists $\tau_0 > 0$ such that $(\mathrm{Fl}^X_\tau, \mathrm{Fl}^Y_\tau)^* w$ is defined on $M \times M$ for all τ with $|\tau| \leq \tau_0$. By Proposition A.2 (2), $\tau \mapsto (\mathrm{Fl}^X_\tau, \mathrm{Fl}^Y_\tau)^* w$ is smooth into $\Gamma_c(M \times M, \mathrm{pr}_2^*\mathrm{T}^s_r M)$ and $\mathrm{L}_{X,Y} w = \frac{\mathrm{d}}{\mathrm{d}\tau}\big|_0 (\mathrm{Fl}^X_\tau, \mathrm{Fl}^Y_\tau)^* w$ in the (LF)-sense. Setting $Y = 0$, it follows that $(\mathrm{L}_{X,0}\, w)^\vee(p) = \frac{\mathrm{d}}{\mathrm{d}\tau}\big|_0 w^\vee(\mathrm{Fl}^X_\tau p)$ in the (LF)-sense in $(\mathcal{T}_r^s)_c(M)$. From this we finally arrive at

$$\begin{aligned}
\mathrm{L}_X \langle v(.), A^s_r(p,.)\,\tilde{t}(p) \otimes \omega(.)\rangle &= \mathrm{L}_X \langle v, w^\vee(p) \otimes \omega\rangle \\
&= \tfrac{\mathrm{d}}{\mathrm{d}\tau}\big|_0 \langle v, w^\vee(\mathrm{Fl}^X_\tau p) \otimes \omega\rangle \\
&= \langle v, \tfrac{\mathrm{d}}{\mathrm{d}\tau}\big|_0 w^\vee(\mathrm{Fl}^X_\tau p) \otimes \omega\rangle \\
&= \langle v, (\mathrm{L}_{X,0}\, w)^\vee(p) \otimes \omega\rangle \\
&= \langle v(.), \mathrm{L}_{X,0}(A^s_r(p,.)\,\tilde{t}(p)) \otimes \omega(.)\rangle.
\end{aligned}$$

For the second term on the right hand side of equation (6.12) we obtain, using the fact that v is linear and continuous.

$$\begin{aligned}
&\mathrm{d}_1\big(\iota^r_s(v)\big)(\omega, p, A)(\mathrm{L}_X\omega)\cdot \tilde{t}(p) \\
&\quad = d\big[\omega \mapsto \langle v(.), A^s_r(p,.)\,\tilde{t}(p) \otimes \omega(.)\rangle\big](\mathrm{L}_X\omega) \\
&\quad = \langle v(.), A^s_r(p,.)\,\tilde{t}(p) \otimes \mathrm{L}_X\omega(.)\rangle \\
&\quad = \langle v(.), \mathrm{L}_{0,X}\big(A^s_r(p,.)\,\tilde{t}(p) \otimes \omega(.)\big)\rangle - \langle v(.), \mathrm{L}_{0,X}\big(A^s_r(p,.)\,\tilde{t}(p)\big) \otimes \omega(.)\rangle \\
&\quad = -\langle \mathrm{L}_X v(.), A^s_r(p,.)\,\tilde{t}(p) \otimes \omega(.)\rangle - \langle v(.), (\mathrm{L}_{0,X} A^s_r)(p,.)\,\tilde{t}(p) \otimes \omega(.)\rangle \\
&\quad = -\iota^r_s(\mathrm{L}_X v)(\omega, p, A)\,\tilde{t}(p) - \langle v(.), (\mathrm{L}_{0,X} A^s_r)(p,.)\,\tilde{t}(p) \otimes \omega(.)\rangle. \qquad (6.14)
\end{aligned}$$

Since $A \mapsto A^s_r$ is the composition of a multilinear map with the diagonal map, the third term on the right hand side of equation (6.12) gives

$$\begin{aligned}
\mathrm{d}_3\big(\iota^r_s(v)\big)(\omega, p, A)&(\mathrm{L}_{X,X}A)\cdot \tilde{t}(p) \\
&= d\big[A \mapsto \langle v(.), A^s_r(p,.)\,\tilde{t}(p) \otimes \omega(.)\rangle\big](\mathrm{L}_{X,X}A) \\
&= \langle v(.), \big(\mathrm{L}_{X,X}(A^s_r)\big)(p,.)\,\tilde{t}(p) \otimes \omega(.)\rangle. \qquad (6.15)
\end{aligned}$$

Combining equations (6.13), (6.14), and (6.15) we obtain the result. □

Standard operations of tensor calculus carry over to elements of $\hat{\mathcal{E}}^r_s(M)$. Thus, for $u_1 \in \hat{\mathcal{E}}^r_s(M)$, $u_2 \in \hat{\mathcal{E}}^{r'}_{s'}(M)$ we define the tensor product $u_1 \otimes u_2$ by

$$(u_1 \otimes u_2)(\omega, p, A) := (u_1(\omega, A) \otimes u_2(\omega, A))(p).$$

Then clearly $u_1 \otimes u_2 \in \hat{\mathcal{E}}^{r+r'}_{s+s'}(M)$. Moreover, if $C^i_j : \mathcal{T}^r_s(M) \to \mathcal{T}^{r-1}_{s-1}(M)$ is any contraction then for $u \in \hat{\mathcal{E}}^r_s(M)$ we define $C^i_j(u) \in \hat{\mathcal{E}}^{r-1}_{s-1}(M)$ by

$$C^i_j(t)(\omega, p, A) := C^i_j(t(\omega, A))(p).$$

Contraction $u_1 \cdot u_2$ of dual fields $u_1 \in \hat{\mathcal{E}}^r_s(M)$ and $u_2 \in \hat{\mathcal{E}}^s_r(M)$ is then defined as a composition of the above operations. Notationally suppressing the embedding σ^r_s, we obtain the special case $u \in \hat{\mathcal{E}}^r_s(M)$, $\tilde{t} \in \mathcal{T}^s_r(M)$:

$$(u \cdot \tilde{t})(\omega, p, A) := (u(\omega, A) \cdot \tilde{t})(p).$$

Proposition 6.9. *The Lie derivative $\hat{\mathrm{L}}_X$ acting on tensor products of arbitrary fields and on contractions of dual fields satisfies the Leibniz rule, i.e. we have*

$$\begin{aligned} \hat{\mathrm{L}}_X(u_1 \otimes u_2) &= (\hat{\mathrm{L}}_X u_1) \otimes u_2 + u_1 \otimes (\hat{\mathrm{L}}_X u_2) \quad (u_1 \in \hat{\mathcal{E}}^r_s(M),\ u_2 \in \hat{\mathcal{E}}^{r'}_{s'}(M)) \\ \hat{\mathrm{L}}_X(u_1 \cdot u_2) &= (\hat{\mathrm{L}}_X u_1) \cdot u_2 \ + \ u_1 \cdot (\hat{\mathrm{L}}_X u_2) \quad (u_1 \in \hat{\mathcal{E}}^r_s(M),\ u_2 \in \hat{\mathcal{E}}^s_r(M)). \end{aligned}$$

Proof. We consider the three defining terms adding up to $\hat{\mathrm{L}}_X$ according to (6.10) separately. As to $\hat{\mathrm{L}}_{0,X,0}$, we have $\hat{\mathrm{L}}_{0,X,0}(u_1 \otimes u_2)(\omega, p, A) = \mathrm{L}_X((u_1 \otimes u_2)(\omega, A))(p) = \mathrm{L}_X(u_1(\omega, A) \otimes u_2(\omega, A))(p)$; for the latter the classical Leibniz rule of course holds. Concerning $\hat{\mathrm{L}}_{X,0,0}$, we note that the corresponding terms are but directional derivatives of the smooth functions $u_1 \otimes u_2$ resp. u_1 resp. u_2. Since the first originates from the remaining two by composition with a bounded bilinear (equivalently, smooth, by [26, 5.5]) map the Leibniz rule holds also in this case. *Mutatis mutandis*, the same arguments apply to $\hat{\mathrm{L}}_{0,0,X}$. The proof for the contraction of dual fields proceeds along the same lines. □

For the proof that $\hat{\mathrm{L}}_X$ respects moderateness resp. negligibility in Section 8 we will need an explicit expression for $\mathrm{d}_3\big((\omega, p, A) \mapsto \mathrm{L}_X(u(\omega, A))(p)\big)(B)$, which will imply that directional derivatives with respect to slots 1 resp. 3 can be interchanged with Lie derivatives with respect to slot 2. We consider the case of slots 2 and 3; the argument for slots 1 and 2 is similar. To this end, let $\phi : \hat{\mathcal{E}}^r_s(M) \to \mathcal{C}^\infty(\hat{\mathcal{A}}_0(M) \times \hat{\mathcal{B}}(M), \mathcal{T}^r_s(M))$ denote the linear isomorphism given by Lemma 7.2. Using ϕ, the map $\hat{\mathrm{L}}_{0,X,0}$ sending $u \in \hat{\mathcal{E}}^r_s(M)$ to $((\omega, p, A) \mapsto \mathrm{L}_X(u(\omega, A))(p))$ can be written as $\hat{\mathrm{L}}_{0,X,0} = \phi^{-1} \circ (\mathrm{L}_X)_* \circ \phi$ where $(\mathrm{L}_X)_*(\tilde{u}) := \mathrm{L}_X \circ \tilde{u}$ for $\tilde{u} \in \mathcal{C}^\infty(\hat{\mathcal{A}}_0(M) \times \hat{\mathcal{B}}(M), \mathcal{T}^r_s(M))$. Recall that $(\mathrm{L}_X)_*$ maps $\mathcal{C}^\infty(\hat{\mathcal{A}}_0(M) \times \hat{\mathcal{B}}(M), \mathcal{T}^r_s(M))$ linearly into itself, due to the boundedness (equivalently, smoothness) of L_X on $\mathcal{T}^r_s(M)$, with respect to the (F)-topology.

Lemma 6.10. *For $B \in \hat{\mathcal{B}}(M)$ let $\mathrm{d}_B : \hat{\mathcal{E}}^r_s(M) \to \hat{\mathcal{E}}^r_s(M)$ denote the directional derivative defined by $(\mathrm{d}_B u)(\omega, p, A) = \mathrm{d}_3 u(\omega, p, A)(B)$. Then for any $X \in \mathfrak{X}(M)$,*

$$\mathrm{d}_B \circ \hat{\mathrm{L}}_{0,X,0} = \hat{\mathrm{L}}_{0,X,0} \circ \mathrm{d}_B,$$

hence, neglecting ϕ,

$$\mathrm{d}_3\big((\omega, p, A) \mapsto \mathrm{L}_X(u(\omega, A))(p)\big)(B) = \mathrm{L}_X(\mathrm{d}_2 u(\omega, A)(B))(p).$$

Proof. For $B \in \hat{\mathcal{B}}(M)$ let $\tilde{\mathrm{d}}_B : \mathcal{C}^\infty(\hat{\mathcal{A}}_0(M) \times \hat{\mathcal{B}}(M), \mathcal{T}^r_s(M)) \to \mathcal{C}^\infty(\hat{\mathcal{A}}_0(M) \times \hat{\mathcal{B}}(M), \mathcal{T}^r_s(M))$ denote the directional derivative defined by $(\tilde{\mathrm{d}}_B \tilde{u})(\omega, A) = \mathrm{d}_2 \tilde{u}(\omega, A)(B)$. We will show the following two relations:

$$\tilde{\mathrm{d}}_B \circ (\mathrm{L}_X)_* = (\mathrm{L}_X)_* \circ \tilde{\mathrm{d}}_B \tag{6.16}$$

$$\phi \circ \mathrm{d}_B = \tilde{\mathrm{d}}_B \circ \phi. \tag{6.17}$$

Transferring (6.16) by means of (6.17) and the defining relation

$$\phi \circ \hat{\mathrm{L}}_{0,X,0} = (\mathrm{L}_X)_* \circ \phi \tag{6.18}$$

from $\mathcal{C}^\infty(\hat{\mathcal{A}}_0(M) \times \hat{\mathcal{B}}(M), \mathcal{T}^r_s(M))$ to $\hat{\mathcal{E}}^r_s(M)$ will accomplish the proof.

(6.16) is a consequence of the chain rule [26, 3.18]: For $\tilde{u} \in \mathcal{C}^\infty(\hat{\mathcal{A}}_0(M) \times \hat{\mathcal{B}}(M), \mathcal{T}^r_s(M))$, we obtain

$$(\tilde{\mathrm{d}}_B \circ (\mathrm{L}_X)_*)(\tilde{u}) = \tilde{\mathrm{d}}_B\,(\mathrm{L}_X \circ \tilde{u}) = \mathrm{L}_X \circ (\tilde{\mathrm{d}}_B\,\tilde{u}) = ((\mathrm{L}_X)_* \circ \tilde{\mathrm{d}}_B)(\tilde{u}).$$

(6.17), on the other hand, follows from the continuity of point evaluations on the (F)-space $\mathcal{T}^r_s(M)$: For $u \in \hat{\mathcal{E}}^r_s(M)$, we have

$$\begin{aligned}
(\tilde{\mathrm{d}}_B\,\phi\,u)(\omega, A)(p) &= \lim_{\tau \to 0} \tfrac{1}{\tau}\big[(\phi\,u)(\omega, A + \tau B) - (\phi\,u)(\omega, A)\big](p) \\
&= \lim_{\tau \to 0} \tfrac{1}{\tau}\big[(\phi\,u)(\omega, A + \tau B)(p) - (\phi\,u)(\omega, A)(p)\big] \\
&= (\mathrm{d}_B\,u)(\omega, p, A) \\
&= (\phi\;\mathrm{d}_B\,u)(\omega, A)(p).
\end{aligned}$$

Note that the first limit above is in the (F)-space $\mathcal{T}^r_s(M)$ while the second one refers to Euclidean topology in $(\mathrm{T}^r_s)_p M \cong (\mathbb{R}^n)^{r+s}$. $\square$

Remark 6.11. To conclude this section we mention without proof some further properties of $\hat{\mathcal{E}}^r_s$.

(i) Since elements of $\hat{\mathcal{A}}_0(M)$ and $\hat{\mathcal{B}}(M)$ are compactly supported, there is an obvious notion of restriction of any $u \in \hat{\mathcal{E}}^r_s(M)$ to open subsets of M.

(ii) For any $v \in \mathcal{D}'^r_s(M)$, $\iota^r_s(v)$ vanishes on the same open subsets of M as v does, hence $\operatorname{supp} \iota^r_s(v)$ is well-defined and equals $\operatorname{supp} v$.

(iii) For coverings $\mathcal{U}$ of M directed by inclusion ($U_1, U_2 \in \mathcal{U} \Rightarrow \exists U_3 \in \mathcal{U}$, $U_3 \supseteq U_1 \cup U_2$) the usual sheaf properties hold for $\hat{\mathcal{E}}^r_s$. A local geometrical definition of the Lie derivative on $\hat{\mathcal{E}}^r_s(M)$ with respect to arbitrary smooth vector fields can be based on this.

7 Smoothness of embedded distributions

The seemingly innocuous statement of $\iota^r_s(v)$ being smooth is, in fact, a deep result involving the entire range of results assembled in Appendix B. The difficulties in proving it reflect the interplay between the apparatus of (smooth as well as distributional) differential geometry and calculus on (infinite-dimensional) locally convex spaces. Observe that in the scalar case treated in [19], the question of smoothness of $\iota(v)$ (for $v \in \mathcal{D}'(M)$) reduces to the trivial statement that $\iota(v) = v \circ \mathrm{pr}_1$, being linear and continuous (hence bounded), is smooth on $\hat{\mathcal{A}}_0(M) \times M$.

The main difficulty becomes clear from the fact that ι^r_s has to bridge the "topology gap" between two worlds: Its argument $v \in \mathcal{D}'^r_s(M)$ relates to the domain of linear spaces carrying (F)- resp. (LF)-topologies and their dual spaces whereas the relevant results on the basic space $\hat{\mathcal{E}}^r_s(M)$ (of which $\iota^r_s(v)$ is a member) hold with respect to the canonical convenient vector space topology on spaces of smooth functions denoted by the term (C)-topology in the sequel (cf. [26]).

One crucial step of the proof consists in getting a handle on the parametrized tensor field $\theta(A, \tilde{t}, p) : q \mapsto A^s_r(p, q)\, \tilde{t}(p)$ occurring in the definition of ι^r_s, the spreading of $\tilde{t}(p)$ (over M) via A (cf. Section 6). Recall that θ has already been used in the proof of Proposition 6.6. Proving the smoothness of $\theta(A, \tilde{t}, p)$ as a function of its three arguments will be accomplished by Lemmata B.7 resp. B.8 and Corollary B.10 of Appendix B, providing the necessary information on spaces of sections of vector bundles over (possibly infinite-dimensional) manifolds. For decomposing θ into manageable parts, we introduce the following terminology:

- For manifolds B' (possibly infinite-dimensional), B, a smooth map $f : B' \to B$ and a vector bundle $E \xrightarrow{\pi} B$ let f^*E denote the pullback bundle of E under f (cf. Appendix A and the discussion following Remark B.8). We define the pullback operator $f^* : \Gamma(B, E) \to \Gamma(B', f^*E)$ as follows: For $u \in \Gamma(B, E)$ given, the pair $(\mathrm{id}_{B'}, u \circ f)$ induces the smooth section $f^*u : B' \to f^*E$, $f^*u = p \mapsto (p, u(f(p)))$ of f^*E.

- For formalizing the spreading process based on the action of a transport operator on tangent vectors we introduce the operator

$$\mathrm{ev}_0^1 : \Gamma(\mathrm{TO}(M,N)) \times \Gamma(\mathrm{pr}_1^*(\mathrm{T}M)) \;\to\; \Gamma(\mathrm{pr}_2^*(\mathrm{T}N))$$

  by defining, for $(p,q) \in M \times N$,

$$(\mathrm{ev}_0^1(A,\tilde{\xi}))(p,q) := \big((p,q)\;,\; A(p,q)\,\bar{\xi}(p,q)\big)$$

  where $A \in \Gamma(\mathrm{TO}(M,N))$ and $\tilde{\xi} \in \Gamma(\mathrm{pr}_1^*(\mathrm{T}M))$ is of the form $(p,q) \mapsto ((p,q), \bar{\xi}(p,q))$. By ev_r^s we denote the obvious extension of ev_0^1 to the (s,r)-case, with A_r^s acting fiberwise on a section $\tilde{\xi} \in \Gamma(\mathrm{pr}_1^*\mathrm{T}_r^s M)$.

- For manifolds M, N and a vector bundle $E \xrightarrow{\pi} N$, we define the operator $\overline{\mathrm{ev}} : \Gamma(M \times N, \mathrm{pr}_2^* E) \times M \to \Gamma(N,E)$ by $\overline{\mathrm{ev}}(u,p)(q) := \mathrm{pr}_2{}'(u(p,q))$ where $\mathrm{pr}_2{}' : \mathrm{pr}_2^* E \to E$ denotes the canonical projection of the pullback bundle $\mathrm{pr}_2^* E$ (cf. Appendix A).

With this notational machinery available, we are able to factorize θ as

$$\theta(A,\tilde{t},p) = A_r^s(p,.)\,\tilde{t}(p) = \overline{\mathrm{ev}}(\mathrm{ev}_r^s(A,\mathrm{pr}_1^*\tilde{t}),p). \tag{7.1}$$

Proposition 7.1 (Smoothness of $\iota_s^r(v)$)**.** *For any $v \in \mathcal{D}'^r_s(M)$ the function $\iota_s^r(v)$ introduced in Definition 6.4 (ii) is smooth, hence a member of $\hat{\mathcal{E}}_s^r(M)$.*

In addition to employing the factorization (7.1), the proof of Proposition 7.1 is based upon an equivalent representation of the basic space $\hat{\mathcal{E}}_s^r(M)$ as a space of smooth four-slot functions taking $\omega, p, A, \tilde{t}$ as arguments. The benefit of such a representation should be clear from Definition 6.4. We abbreviate the property of being "$\mathcal{C}^\infty(M)$-linear in the k-th slot" as being "$\mathcal{C}_k$-linear".

Lemma 7.2. *The basic space $\hat{\mathcal{E}}_s^r(M)$ has the following equivalent representations which are mutually isomorphic as linear spaces:*

(0) $\{u \in \mathcal{C}^\infty(\hat{\mathcal{A}}_0(M) \times M \times \hat{\mathcal{B}}(M)\,,\,\mathrm{T}_s^r M) \mid u(\omega,p,A) \in (\mathrm{T}_s^r)_p M\}$

(1) $\Gamma(\hat{\mathcal{A}}_0(M) \times M \times \hat{\mathcal{B}}(M), \mathrm{pr}_2^*\mathrm{T}_s^r M)$

(2) $\mathcal{C}^\infty(\hat{\mathcal{A}}_0(M) \times \hat{\mathcal{B}}(M)\,,\,\mathcal{T}_s^r(M))$

(3) $\{u \in \mathcal{C}^\infty(\hat{\mathcal{A}}_0(M) \times \hat{\mathcal{B}}(M)\,,\,\mathcal{C}^\infty(\mathcal{T}_r^s(M), \mathcal{C}^\infty(M))) \mid u(\omega,A)$ *is* $\mathcal{C}_1$*-linear*$\}$

(4) $\{u \in \mathcal{C}^\infty(\hat{\mathcal{A}}_0(M) \times \hat{\mathcal{B}}(M) \times \mathcal{T}_r^s(M) \times M\,,\,\mathbb{R}) \mid u$ *is* $\mathcal{C}_3$*-linear*$\}$

(5) $\{u \in \mathcal{C}^\infty(\hat{\mathcal{A}}_0(M)\times M\times\hat{\mathcal{B}}(M)\,,\,\mathcal{C}^\infty(\mathcal{T}^s_r(M),\mathbb{R}))\mid u(\omega,p,A)$ *is* $\mathcal{C}_1$*-linear*$\}$.

The relations between corresponding elements $t^{[i]}$ *from space (i), respectively, of the above list are given by*

$$\begin{aligned}((\omega,p,A),u^{[0]}(\omega,p,A)) &= u^{[1]}(\omega,p,A)\\ u^{[0]}(\omega,p,A) &= u^{[2]}(\omega,A)(p)\\ u^{[2]}(\omega,A)\cdot\tilde{t} &= u^{[3]}(\omega,A)(\tilde{t})\\ u^{[3]}(\omega,A)(\tilde{t})(p) &= u^{[4]}(\omega,A,\tilde{t},p) = u^{[5]}(\omega,p,A)(\tilde{t})\end{aligned}$$

where $\omega\in\hat{\mathcal{A}}_0(M)$*,* $p\in M$*,* $A\in\hat{\mathcal{B}}(M)$ *and* $\tilde{t}\in\mathcal{T}^s_r(M)$*.*

Remark 7.3. (i) Note that, even for finite-dimensional M, (1) requires a theory of vector bundles over infinite-dimensional smooth manifolds (in fact, over $\hat{\mathcal{A}}_0(M)\times M\times\hat{\mathcal{B}}(M)$ in the case at hand); see the remarks preceding Lemma B.9 in Appendix B.

(ii) In order to give meaning to the various notions of smoothness occurring in (0)–(5) of the preceding lemma, we have to specify appropriate locally convex topologies resp. bornologies on the spaces involved. To this end, we equip $\hat{\mathcal{A}}_0(M)$ and $\hat{\mathcal{B}}(M)$ with their respective (LF)-topologies and $\mathcal{T}^s_r(M)$ with its usual (F)-topology (recall our convention stated in Section 2 for M non-separable). On the other hand, whenever a space of smooth functions such as $\mathcal{C}^\infty(.,..)$ or $\mathcal{T}^r_s(M)$ appears at the second slot of some $\mathcal{C}^\infty(...,....)$ it carries the locally convex topology (C) defined in Appendix B. This is indispensable for legitimizing the applications of [26, 27.17] resp. of Lemma B.9 which are to follow. Whenever an explicit declaration of the topology in question is needed we will use subscripts as, e.g., in $\mathcal{C}^\infty(M)_F$ resp. $\mathcal{C}^\infty(M)_C$.

(iii) In (4), $\mathcal{C}_3$-linearity (resp. $\mathcal{C}_1$-linearity in (5)), in fact, are to be understood as

$$u(\omega,A,f\cdot\tilde{t},p) := u(\omega,A,\tilde{t},p)\cdot f(p)$$

resp.

$$u(\omega,p,A)(f\cdot\tilde{t}) := u(\omega,p,A)(\tilde{t})\cdot f(p)$$

with $f\in\mathcal{C}^\infty(M)$, in order to guarantee compatibility with (3) which, in turn could also be written as $\mathcal{C}^\infty(\hat{\mathcal{A}}_0(M)\times\hat{\mathcal{B}}(M)\,,\,\mathrm{L}^b_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M),\mathcal{C}^\infty(M)))$ where $\mathrm{L}^b_{\mathcal{C}^\infty(M)}(.,..)$ denotes the subspace of $\mathcal{C}^\infty(.,..)$ of $\mathcal{C}^\infty(M)$-linear bounded (hence smooth, cf. [26, 2.11]) functions. [26, 5.3] shows that this does not cause any ambiguity as to the meaning of a subset of $\mathrm{L}^b_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M),\mathcal{C}^\infty(M))$ being (C)-bounded and, hence, of a mapping into that space being smooth.

(iv) (0) is precisely the expression for $\hat{\mathcal{E}}^r_s(M)$ given in Definition 6.2; (2) corresponds to writing $u(\omega,A)(p)=u(\omega,p,A)$, introduced after Definition 6.2. As to the other items, cf. Remark 7.4.

Proof of Lemma 7.2. The form of the pullback bundle $\mathrm{pr}_2^*\mathrm{T}^r_s M$ occurring in (1) shows that its smooth sections are precisely given by maps as in (0) (compare also the discussion preceding Corollary B.10), the correspondence being as stated in the lemma. The equality of (1) and (2) as well as the relation $u^{[1]}(\omega, p, A) = ((\omega, p, A), u^{[2]}(\omega, A)(p))$ are immediate from Lemma B.9. Moving on to (2)–(5), it is clear from the given relations that passing from $u^{[i]}$ to $u^{[i+1]}$ resp. vice versa yields maps having appropriate domains, ranges (disregarding smoothness) and algebraic properties for the cases $i = 3, 4$. From [26, 27.17] we conclude that $u^{[4]}$ is smooth if and only if $u^{[5]}$ is smooth and takes *smooth* functions on $\mathcal{T}^s_r(M)$ as values on triples (ω, p, A), due to $u^{[5]} = (u^{[4]})^\vee$ in the terminology of [26, 3.12], with respect to the variable $\tilde{t}$ getting separated from the variables ω, p, A. A twofold application of the same argument achieves the transfer of smoothness between $u^{[3]}$ and $u^{[4]}$.

It remains to discuss $i = 2$. Observe that the assignment $t \mapsto (\tilde{t} \mapsto t \cdot \tilde{t})$ embeds $\mathcal{T}^r_s(M)$ into the space of $\mathcal{C}^\infty(M)$-linear bounded (hence smooth) maps from $\mathcal{T}^s_r(M)_F$ into $\mathcal{C}^\infty(M)_F$. (C) being weaker than the (F)-topology on $\mathcal{C}^\infty(M)$, we obtain smoothness from $\mathcal{T}^s_r(M)_F$ into $\mathcal{C}^\infty(M)_C$, as required for (3). Now we have the chain of inclusions

$$\mathcal{T}^r_s(M) \subseteq \mathrm{L}^b_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M)_F, \mathcal{C}^\infty(M)_C) \subseteq \mathrm{L}_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{C}^\infty(M)),$$

where $\mathrm{L}_{\mathcal{C}^\infty(M)}(.,..)$ denotes the respective space of *all* $\mathcal{C}^\infty(M)$-linear maps. $\mathcal{T}^r_s(M)$ being isomorphic to $\mathrm{L}_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{C}^\infty(M))$ via the assignment specified above, all three spaces in the chain are, in fact, identical. Moreover, by Theorem B.5 of Appendix B, the corresponding (C)-topologies on $\mathcal{T}^r_s(M)$ and $\mathrm{L}^b_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{C}^\infty(M))$ have the same bounded sets, showing that also the spaces given by (2) and (3) are identical. $\square$

Remark 7.4. Observe that each of (0)–(5) serves a distinct yet prominent purpose: (0) was used in introducing the basic space (cf. Definition 6.2), with a view to representing best the intuitive picture of a (representative of a) generalized tensor field as a section of $\mathrm{T}^r_s M$ depending on additional parameters ω, A. The drawback of (0) consists in the fact that the range space $\mathrm{T}^r_s M$ is not a linear space, hence $\mathcal{C}^\infty(\hat{\mathcal{A}}_0(M) \times M \times \hat{\mathcal{B}}(M)\,,\, \mathrm{T}^r_s M)$ is not a (convenient) vector space. Making up for this deficiency, (1) opens the gates to applying the apparatus of infinite dimensional differential geometry as provided by [26, Sec. 27–30]. In particular, it paves the way to (2)–(5): (2) is optimal for defining σ^r_s by $\sigma^r_s(t)(\omega, A) := t$ (cf. Definition 6.4 (i)). (3) provides the crucial intermediate step bringing $\tilde{t}$ into play. Concerning an explicit definition of ι^r_s, (0)–(2) are flawed by not providing a slot for inserting $\tilde{t}$. Among (3)–(5), (5) (which, indeed, was used in (ii) of Definition 6.4) seems to be the optimal choice due to having ω, p, A as primary arguments and $\tilde{t}$

to be acted upon, yet also (3) and (4) are capable of doing the job. As to proving smoothness of $\iota_s^r(v)$, finally, (0) resp. (1) resp. (4) are to be preferred, relying exclusively on the well-known (F)- resp. (LF)-topologies. Altogether, for establishing smoothness of $\iota_s^r(v)$, (4) turns out to be the best choice.

Now, having Lemma 7.2 at our disposal, we shall demonstrate that $\iota_s^r(v)$ is smooth.

Proof of Proposition 7.1. By Lemma 7.2 it suffices to show that for $v \in \mathcal{D}'^r_s(M)$, $\iota_s^r(v)$ is a member of (4) as defined in Lemma 7.2. The crucial step of the proof consists in establishing $\theta : (A, \tilde{t}, p) \mapsto (q \mapsto A_r^s(p,q)\, \tilde{t}(p))$ to be smooth as a map $\theta : \hat{\mathcal{B}}(M)_{LF} \times \mathcal{T}_r^s(M)_F \times M \to \mathcal{T}_r^s(M)_F$. Once this has been achieved, it suffices to note that the bilinear map

$$\mathcal{T}_r^s(M)_F \times \Omega^n_{\mathrm{c}}(M)_{LF} \ni (\tilde{\tilde{t}}, \omega) \mapsto \tilde{\tilde{t}} \otimes \omega \in (\mathcal{T}_r^s(M) \otimes_{\mathcal{C}^\infty(M)} \Omega^n_{\mathrm{c}}(M))_{LF}$$

is bounded, hence smooth (which is immediate from an inspection of the seminorms defining the (F)- resp. (LF)-topologies on section spaces of vector bundles, cf. (2.1)) to conclude, finally, that

$$\iota_s^r(v)^{[4]}(\omega, A, \tilde{t}, p) = \langle v, \theta(A, \tilde{t}, p) \otimes \omega \rangle$$

is a smooth function on $\hat{\mathcal{A}}_0(M)_{LF} \times \hat{\mathcal{B}}(M)_{LF} \times \mathcal{T}_r^s(M)_F \times M$, due to the continuity (hence boundedness, hence smoothness) of v.

To see the smoothness of θ, we write, using (7.1),

$$\theta(A, \tilde{t}, p) = \overline{\mathrm{ev}}(\mathrm{ev}_r^s(A, \mathrm{pr}_1^* \tilde{t}), p).$$

By Lemmata B.7 and B.8 we obtain continuity (hence boundedness resp. smoothness) of $(A, \tilde{t}) \mapsto (A, \mathrm{pr}_1^* \tilde{t}) \mapsto \mathrm{ev}_r^s(A, \mathrm{pr}_1^* \tilde{t})$ with respect to the (LF)- resp. (F)-topologies, while Corollary B.10 yields smoothness of $(\mathrm{ev}_r^s(A, \mathrm{pr}_1^* \tilde{t}), p) \mapsto \overline{\mathrm{ev}}(\mathrm{ev}_r^s(A, \mathrm{pr}_1^* \tilde{t}), p) = \theta(A, \tilde{t}, p)$ with respect to the (C)-topologies on the section spaces. (C) being weaker than (F), we can combine both smoothness statements to obtain the smoothness of θ with respect to the (C)-topology on the target space $\mathcal{T}_r^s(M)$. Finally, Corollary B.2 permits us to replace the (C)-topology on $\mathcal{T}_r^s(M)$ by the (F)-topology. $\square$

Remark 7.5. Based upon a modification of the map θ employed above, it is possible to arrive at a definition of the embedding $\iota_s^r : \mathcal{D}'^r_s(M) \to \hat{\mathcal{E}}_s^r(M)$ not explicitly containing $\tilde{t} \in \mathcal{T}_r^s(M)$: Using results of [15], every distribution $v \in \mathcal{D}'^r_s(M)$ can be represented as a bounded resp. continuous $\mathcal{C}^\infty(M)$-linear map $v^\vee$ from $\Omega^n_{\mathrm{c}}(M)$ into $\mathcal{T}_r^s(M)'$, the topological dual of the (F)-space $\mathcal{T}_r^s(M)$. The relation between v and $v^\vee$ is given by $v(\tilde{t} \otimes \omega) = \langle v^\vee(\omega), \tilde{t} \rangle$, for $\omega \in$

$\Omega^n_c(M)$ and $\tilde{t} \in \mathcal{T}^s_r(M)$. Introducing the spreading operator spr : $\hat{\mathcal{B}}(M) \times M \to \mathrm{L}^b(\mathcal{T}^s_r(M), \mathcal{T}^s_r(M))$ by

$$\mathrm{spr}(A,p)(\tilde{t}) := \theta(A, \tilde{t}, p)$$

we obtain $\iota^r_s(v)(\omega, p, A) = v^\vee(\omega) \circ \mathrm{spr}(A,p)$ (where $\iota^r_s(v)$ is to be understood as $\iota^r_s(v)^{[5]}$). Smoothness of spr follows from smoothness of θ (cf. the proof of Proposition 7.1) via the exponential law in [26, 27.17].

8 Dynamics

We now turn to the analytic core of our approach: the quotient construction of tensor algebras of generalized functions displaying maximal compatibility properties with respect to smooth and distributional tensor fields. As in the scalar case ([19], see Section 3) our approach is based on singling out subspaces of moderate resp. negligible maps in the basic space $\hat{\mathcal{E}}^r_s(M)$ by requiring asymptotic estimates of the derivatives of representatives when evaluated along smoothing kernels (Definition 3.4). We thereby adhere to the basic strategy ([17, Ch. 9]) of separating the basic definitions (the kinematics, in our current terminology) from the testing (of the asymptotic estimates underlying the quotient construction of the space of generalized tensor fields, or, for short, the dynamics). Since the representatives of generalized tensors depend not only on points $p \in M$ and n-forms ω as in the scalar case [19] but also on transport operators A, a new feature of the following construction is that derivatives with respect to A will have to be taken into account as well.

The notion of the core of a transport operator plays an important role in our construction (in particular, in Lemma 8.6 below and its applications). In what follows, for any $U \subseteq M$, U° denotes the interior of U.

Definition 8.1. For any transport operator $A \in \hat{\mathcal{B}}(M)$ we define the core of A by

$$\mathrm{core}(A) := \{p \in M \mid A(p,p) = \mathrm{id}_{T_pM}\}^\circ.$$

Remark 8.2. Given $K \subset\subset M$ there always exists some $A \in \hat{\mathcal{B}}(M)$ with $K \subseteq \mathrm{core}(A)$. Clearly such an A can be obtained by gluing together local identity matrices. For a more geometrical approach, choose any Riemannian metric g on M and denote by $r(p)$ the injectivity radius at p with respect to g. On the open neighborhood $W := \{(p,q) \mid q \in B_{r(p)}(p)\}$ of the diagonal in $M \times M$ we define a transport operator A' by letting $A'(p,q)$ be parallel transport along the unique radial geodesic in $B_{r(p)}(p)$ from p to q. Now choose some $\chi \in \mathcal{D}(W)$ with $\chi(p,p) = 1$ for all p in a neighborhood of K. Then we may set $A := \chi A'$.

Definition 8.3. For $A \in \hat{\mathcal{B}}(M)$ we define the kernel of A by

$$\ker(A) := \{p \in M \mid A(p,p) = 0\}.$$

If $U \subseteq M$ we set

$$\hat{\mathcal{B}}_U(M) := \{A \in \hat{\mathcal{B}}(M) \mid U \subseteq \ker(A)\}.$$

Note that for any $A \in \hat{\mathcal{B}}(M)$, $\mathrm{L}_X A \in \hat{\mathcal{B}}_{\mathrm{core}(A)}(M)$ (which will be used in Lemma 8.13). Based on these notions we are now ready to introduce the basic building blocks of our construction:

Definition 8.4. An element $u \in \hat{\mathcal{E}}^r_s(M)$ is called moderate if it satisfies the following asymptotic estimates:
$\forall K \subset\subset M\ \forall A \in \hat{\mathcal{B}}(M)$ with $K \subset\subset \mathrm{core}(A)$
$\forall j \in \mathbb{N}\ \forall B_1, \dots, B_j \in \hat{\mathcal{B}}_{\mathrm{core}(A)}(M)$
$\forall l \in \mathbb{N}_0\ \exists N \in \mathbb{N}_0\ \forall X_1, \dots, X_l \in \mathfrak{X}(M)\ \forall \Phi \in \tilde{\mathcal{A}}_0(M)$:

$$\sup_{p \in K} \|\mathrm{L}_{X_1} \dots \mathrm{L}_{X_l}(\mathrm{d}_3^j u(\Phi_{\varepsilon,p}, p, A)(B_1, \dots, B_j))\|_h = O(\varepsilon^{-N}) \qquad (\varepsilon \to 0).$$

The space of moderate tensor fields is denoted by $(\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M)$.

In this definition, $\| \, \|_h$ denotes the norm induced on the fibers of $\mathrm{T}^r_s M$ by any Riemannian metric (changing h does not affect the asymptotic estimates). In the case $r = s = 0$, $\| \, \|_h$ is to be replaced by the absolute value in $\mathbb{R}$.

Definition 8.5. An element $u \in (\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M)$ is called negligible if
$\forall K \subset\subset M\ \forall A \in \hat{\mathcal{B}}(M)$ with $K \subset\subset \mathrm{core}(A)$
$\forall j \in \mathbb{N}\ \forall B_1, \dots, B_j \in \hat{\mathcal{B}}_{\mathrm{core}(A)}(M)$
$\forall l \in \mathbb{N}_0\ \forall m \in \mathbb{N}_0\ \exists k \in \mathbb{N}_0\ \forall X_1, \dots, X_l \in \mathfrak{X}(M)\ \forall \Phi \in \tilde{\mathcal{A}}_k(M)$:

$$\sup_{p \in K} \|\mathrm{L}_{X_1} \dots \mathrm{L}_{X_l}(\mathrm{d}_3^j u(\Phi_{\varepsilon,p}, p, A)(B_1, \dots, B_j))\|_h = O(\varepsilon^{m}) \qquad (\varepsilon \to 0).$$

The space of negligible tensor fields is denoted by $\hat{\mathcal{N}}^r_s(M)$.

The Lie derivatives in the asymptotic estimates in Definitions 8.4 and 8.5 are to be understood as $\mathrm{L}_{X_1} \dots \mathrm{L}_{X_l}$ acting on the smooth section $p \mapsto \mathrm{d}_3^j u(\Phi_{\varepsilon,p}, p, A)(B_1, \dots, B_j)$ of the vector bundle $\mathrm{T}^r_s M$. The fact that the B_i in Definitions 8.4 and 8.5 are supposed to belong to $\hat{\mathcal{B}}_{\mathrm{core}(A)}(M)$ signifies their role as “tangent vectors” when differentiating with respect to A.

Our first aim is to explore the relation between the “scalar” spaces $(\hat{\mathcal{E}}^0_0)_{\mathrm{m}}(M)$, $\hat{\mathcal{N}}^0_0(M)$ and their counterparts $\hat{\mathcal{E}}_{\mathrm{m}}(M)$ and $\hat{\mathcal{N}}(M)$ from Definition 3.5. The following basic lemma introduces a reduction principle that will be referred to repeatedly in what follows.

Lemma 8.6. *(Reduction) Let $u \in \hat{\mathcal{E}}^0_0(M)$. Then for each $j \in \mathbb{N}_0$ and each $(A, B_1, \ldots, B_j) \in \hat{\mathcal{B}}(M)^{j+1}$, the map*

$$(\omega, p) \mapsto \mathrm{d}_3^j u(\omega, p, A)(B_1, \ldots, B_j)$$

is a member of $\hat{\mathcal{E}}(M)$. It is also an element of $\hat{\mathcal{E}}(\mathrm{core}(A))$, when restricted accordingly.
Moreover, $u \in (\hat{\mathcal{E}}^0_0)_{\mathrm{m}}(M)$ if and only if for all $j \in \mathbb{N}_0$, for all $A \in \hat{\mathcal{B}}(M)$ and all $B_1, \ldots, B_j \in \hat{\mathcal{B}}_{\mathrm{core}(A)}(M)$ we have

$$(\omega, p) \mapsto \mathrm{d}_3^j u(\omega, p, A)(B_1, \ldots, B_j) \in \hat{\mathcal{E}}_{\mathrm{m}}(\mathrm{core}(A)).$$

Analogous statements hold for $\hat{\mathcal{N}}^0_0(M)$ and $\hat{\mathcal{N}}(M)$.

Proof. This is immediate by inspecting Definitions 8.4, 8.5 above and the corresponding Definitions 3.5 (i) and (ii). □

As a first consequence we retain the important fact that negligibility for elements of $(\hat{\mathcal{E}}^0_0)_{\mathrm{m}}(M)$ can be characterized without resorting to derivatives with respect to slots 1 and 2:

Corollary 8.7. *Let $u \in (\hat{\mathcal{E}}^0_0)_{\mathrm{m}}(M)$. Then $u \in \hat{\mathcal{N}}^0_0(M)$ if and only if*
$\forall K \subset\subset M$ $\forall A \in \hat{\mathcal{B}}(M)$ with $K \subset\subset \mathrm{core}(A)$
$\forall j \in \mathbb{N}$ $\forall B_1, \ldots, B_j \in \hat{\mathcal{B}}_{\mathrm{core}(A)}$
$\forall m \in \mathbb{N}_0$ $\exists k \in \mathbb{N}_0$ $\forall \Phi \in \tilde{\mathcal{A}}_k(M)$:

$$\sup_{p \in K} \|\mathrm{d}_3^j u(\Phi_{\varepsilon,p}, p, A)(B_1, \ldots, B_j)\|_h = O(\varepsilon^m) \qquad (\varepsilon \to 0).$$

Proof. This follows from Lemma 8.6 and [19, Cor. 4.5]. □

Next, in order to exploit these relations also for general r and s, we introduce a "saturation principle" which characterizes moderateness and negligibility of general tensor fields in terms of scalar fields obtained by saturating (r, s)-tensor fields with dual (smooth) (s, r)-fields.

Proposition 8.8. *(Saturation) Let $u \in \hat{\mathcal{E}}^r_s(M)$. The following are equivalent:*

(i) $u \in (\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M)$.

(ii) *For all $\tilde{t} \in \mathcal{T}^s_r(M)$, $u \cdot \tilde{t} \in (\hat{\mathcal{E}}^0_0)_{\mathrm{m}}(M)$.*

An analogous statement holds for $\hat{\mathcal{N}}^r_s(M)$ and $\hat{\mathcal{N}}^0_0(M)$.

Proof. It will suffice to prove the equivalence in the moderateness case.

(i)⇒(ii): Given $u \in (\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M)$ and $\tilde{t} \in \mathcal{T}^s_r(M)$, let $K \subset\subset \operatorname{core}(A)$, $A \in \hat{\mathcal{B}}(M)$, l, $j \in \mathbb{N}_0$, and $B_1, \dots, B_j \in \hat{\mathcal{B}}_{\operatorname{core}(A)}(M)$. Let $X_1, \dots, X_l \in \mathfrak{X}(M)$, $\Phi \in \tilde{\mathcal{A}}_0(M)$. Then

$$\mathrm{L}_{X_1} \dots \mathrm{L}_{X_l} \mathrm{d}_3^j[(u \cdot \tilde{t})(\Phi_{\varepsilon,p}, p, A)](B_1, \dots, B_j)$$

is a sum of terms of the form

$$\mathrm{L}_{X_{i_1}} \dots \mathrm{L}_{X_{i_k}} \mathrm{d}_3^j u(\Phi_{\varepsilon,p}, p, A)(B_1, \dots, B_j) \cdot \mathrm{L}_{X_{i_{k+1}}} \dots \mathrm{L}_{X_{i_l}} \tilde{t}(p).$$

Here (on K) the first factor is bounded by some ε^{-N}, and the second is bounded independently of ε.

(ii)⇒(i): By induction over l, we deduce the following from (ii): For given $\tilde{t} \in \mathcal{T}^s_r(M)$, $A \in \hat{\mathcal{B}}(M)$ such that $K \subset\subset \operatorname{core}(A)$ and $B_1, \dots, B_j \in \hat{\mathcal{B}}_{\operatorname{core}(A)}(M)$, as well as $l \in \mathbb{N}_0$ there exists some N (depending on $\tilde{t}$) such that for all $X_1, \dots, X_l \in \mathfrak{X}(M)$

$$\sup_{p \in K} \left| \left(\mathrm{L}_{X_1} \dots \mathrm{L}_{X_l} \mathrm{d}_3^j u(\Phi_{\varepsilon,p}, p, A)(B_1, \dots, B_j) \right) \cdot \tilde{t}(p) \right| = O(\varepsilon^{-N}). \qquad (8.1)$$

Indeed, for $l = 0$ the assertion is immediate from (ii) and induction proceeds by the Leibniz rule. We now note that in order to establish moderateness of u, we may additionally suppose in the above that K is contained in some chart neighborhood U. Choose some $\chi \in \mathcal{D}(U)$ which equals 1 in a neighborhood of K and let $\{\tilde{t}_i \mid 1 \le i \le n^{r+s}\}$ be a local basis of $\mathcal{T}^s_r(U)$. Then inserting each $\chi \tilde{t}_i$ into (8.1) and choosing the maximum of the resulting powers ε^{-N_i}, we obtain the desired moderateness estimate for u on K. □

The above saturation principle allows one to extend the validity of Corollary 8.7 to general tensor fields:

Theorem 8.9. *Let $u \in (\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M)$. Then $u \in \hat{\mathcal{N}}^r_s(M)$ if and only if*
$\forall K \subset\subset M$ $\forall A \in \hat{\mathcal{B}}(M)$ *with* $K \subset\subset \operatorname{core}(A)$
$\forall j \in \mathbb{N}$ $\forall B_1, \dots, B_j \in \hat{\mathcal{B}}_{\operatorname{core}(A)}(M)$
$\forall m \in \mathbb{N}_0$ $\exists k \in \mathbb{N}_0$ $\forall \Phi \in \tilde{\mathcal{A}}_k(M)$*:*

$$\sup_{p \in K} \| \mathrm{d}_3^j u(\Phi_{\varepsilon,p}, p, A)(B_1, \dots, B_j) \|_h = O(\varepsilon^m) \qquad (\varepsilon \to 0).$$

Proof. Suppose that $u \in (\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M)$ satisfies the above condition. By Proposition 8.8, for all $\tilde{t} \in \mathcal{T}^s_r(M)$, $u \cdot \tilde{t}$ is a member of $(\hat{\mathcal{E}}^0_0)_{\mathrm{m}}(M)$ and satisfies the negligibility estimates of order zero specified in Corollary 8.7. It is therefore in $\hat{\mathcal{N}}^0_0(M)$ and the claim follows again from Proposition 8.8. □

The following result gives a local characterization of moderateness and negligibility.

Proposition 8.10. *Let $u \in \hat{\mathcal{E}}^r_s(M)$ and let $\mathcal{U}$ be an open cover of M. The following are equivalent:*

(i) $u \in (\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M)$.

(ii) $\forall U \in \mathcal{U}\ \forall K \subset\subset U\ \forall A \in \hat{\mathcal{B}}(M)$ *with* $K \subseteq \mathrm{core}(A)\ \forall j,\ l\ \forall B_1, \dots, B_j \in \hat{\mathcal{B}}_{\mathrm{core}(A)}(M)\ \exists N\ \forall X_1, \dots, X_l \in \mathfrak{X}(U)\ \forall \Phi \in \tilde{\mathcal{A}}_0(U)$:

$$\sup_{p \in K} \|\mathrm{L}_{X_1} \dots \mathrm{L}_{X_l} \mathrm{d}_3^j u(\Phi_{\varepsilon,p}, p, A)(B_1, \dots, B_j)\|_h = O(\varepsilon^{-N}) \qquad (\varepsilon \to 0)\,.$$

An analogous result holds for the estimates defining $\hat{\mathcal{N}}^r_s(M)$.

Proof. Again it will suffice to carry out the proof in the moderateness case.

(i)$\Rightarrow$(ii): Using suitable cut-off functions we may extend the X_i to global vector fields $\tilde{X}_i$ on M which coincide with X_i on a neighborhood of K ($1 \leq i \leq l$). Moreover, given $\Phi \in \tilde{\mathcal{A}}_0(U)$, we pick any $\Psi \in \tilde{\mathcal{A}}_0(M)$ and $\chi \in \mathcal{D}(U)$ with $\chi = 1$ in a neighborhood of K, and set

$$\tilde{\Phi}(\varepsilon, p) := \chi(p)\Phi(\varepsilon, p) + (1 - \chi(p))\Psi(\varepsilon, p).$$

Then $\tilde{\Phi} \in \tilde{\mathcal{A}}_0(M)$ and $\tilde{\Phi}(\varepsilon, p) = \Phi(\varepsilon, p)$ for all p in a neighborhood of K and all $\varepsilon \in (0, 1]$. The moderateness estimate of u with respect to K, $\tilde{X}_1, \dots, \tilde{X}_l$ and $\tilde{\Phi}$ then establishes (ii).

(ii)$\Rightarrow$(i): Let $K \subset\subset M$, $A \in \hat{\mathcal{B}}(M)$, $K \subseteq \mathrm{core}(A)$, j, l, and $B_1, \dots, B_j \in \hat{\mathcal{B}}_{\mathrm{core}(A)}(M)$ be given. Without loss of generality we may suppose that $K \subset\subset U$ for some $U \in \mathcal{U}$. For this set of data we obtain N from (ii). Taking $\tilde{X}_1, \dots, \tilde{X}_l \in \mathfrak{X}(M)$, we set $X_i := \tilde{X}_i|_U$ for $1 \leq i \leq l$. Let $\tilde{\Phi} \in \tilde{\mathcal{A}}_0(M)$. Our aim is to construct $\Phi \in \tilde{\mathcal{A}}_0(U)$ such that $\Phi(\varepsilon, p) = \tilde{\Phi}(\varepsilon, p)$ for p in a neighborhood of K and ε sufficiently small. Thus let W be a relatively compact neighborhood of K in U and choose $\chi \in \mathcal{D}(M)$ with $\mathrm{supp}\,\chi \subseteq U$ and $\chi = 1$ on W. Since $\tilde{\Phi}$ is a smoothing kernel (Definition 3.4) there exist C, $\varepsilon_0 > 0$ such that for all $p \in \mathrm{supp}\,\chi$ and all $\varepsilon \leq \varepsilon_0$ we have $\mathrm{supp}\,\tilde{\Phi}(\varepsilon, p) \subseteq B_{\varepsilon C}(p) \subseteq U$. Choose $\lambda \in \mathcal{C}^\infty(\mathbb{R}, I)$ such that $\lambda = 1$ on $(-\infty, \varepsilon_0/3]$ and $\lambda = 0$ on $[\varepsilon_0/2, \infty)$. Finally, pick any $\Phi_1 \in \tilde{\mathcal{A}}_0(U)$ and set

$$\begin{aligned} \Phi : I \times U &\rightarrow \hat{\mathcal{A}}_0(M) \\ \Phi(\varepsilon, p) &:= (1 - \chi(p)\lambda(\varepsilon))\Phi_1(\varepsilon, p) + \chi(p)\lambda(\varepsilon)\tilde{\Phi}(\varepsilon, p). \end{aligned}$$

It is then easily checked that in fact $\Phi \in \tilde{\mathcal{A}}_0(U)$ and that for $p \in W$ and $\varepsilon \leq \varepsilon_0/3$ we have $\Phi(\varepsilon, p) = \tilde{\Phi}(\varepsilon, p)$. Thus the moderateness test (ii) with

data K, A, j, l, $B_1, \dots, B_j$, $X_1, \dots, X_l$ and Φ gives the desired $(\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M)$-estimate for the same data set, yet with $\tilde{X}_1, \dots, \tilde{X}_l$, $\tilde{\Phi}$ replacing $X_1, \dots, X_l$, Φ. $\square$

Remark 8.11. Note that in the previous result, the transport operators employed in the local tests on the open sets U are supposed to be *global* operators, defined on all of M. Nevertheless, if $\mathcal{U}$ is directed by inclusion as in Remark 6.11 (iii), then

$$u \in (\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M) \Leftrightarrow u|_U \in (\hat{\mathcal{E}}^r_s)_m(U) \ \forall U \in \mathcal{U},$$

and analogously for $\hat{\mathcal{N}}^r_s$.

We are now in a position to establish the main properties of the embeddings ι^r_s and σ^r_s (cf. Def. 6.4). The following result corresponds to **(T1)** in the general scheme of construction introduced in [17, Ch. 3].

Theorem 8.12.

(i) $\iota^r_s(\mathcal{D}'^r_s(M)) \subseteq (\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M)$.

(ii) $\sigma^r_s(\mathcal{T}^r_s(M)) \subseteq (\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M)$.

(iii) $(\iota^r_s - \sigma^r_s)(\mathcal{T}^r_s(M)) \subseteq \hat{\mathcal{N}}^r_s(M)$.

(iv) *If* $v \in \mathcal{D}'^r_s(M)$ *and* $\iota^r_s(v) \in \hat{\mathcal{N}}^r_s(M)$, *then* $v = 0$.

Proof. (i) Let $v \in \mathcal{D}'^r_s(M)$. By saturation (Proposition 8.8) it suffices to show that for each $\tilde{t} \in \mathcal{T}^s_r(M)$, $\iota^r_s(v) \cdot \tilde{t} \in (\hat{\mathcal{E}}^0_0)_{\mathrm{m}}(M)$. In order to verify the moderateness estimates from Definition 8.4, we first consider the case $j = 0$ and $l = 1$. For this, we have

$$\begin{aligned}
&\mathrm{L}_X(\iota^r_s(v)(\Phi(\varepsilon, p), p, A) \cdot \tilde{t}(p)) \\
&\quad = \mathrm{L}_X \langle v, A^s_r(p, \,.\,)\, \tilde{t}(p) \otimes \Phi(\varepsilon, p)(\,.\,) \rangle \\
&\quad = \langle v, \mathrm{L}_{X,0}(A^s_r(p, \,.\,)\, \tilde{t}(p)) \otimes \Phi(\varepsilon, p)(\,.\,) \rangle + \langle v, A^s_r(p, \,.\,)\, \tilde{t}(p) \otimes \mathrm{L}_{X,0}\Phi(\varepsilon, p)(\,.\,) \rangle.
\end{aligned}$$

Here we are viewing $\Phi(\varepsilon, p)(.)$ as a smooth function of $(p, .)$ on $M \times M$ (legitimized by Lemma B.9), enabling us to apply $\mathrm{L}_{X,0}$ similar to the case of $A^s_r(p, \,.\,)\, \tilde{t}(p)$ (cf. the proof of Proposition 6.8). The transition from $\mathrm{L}_X \circ v$ to $v \circ \mathrm{L}_{X,0}$ in the preceding calculation has been argued in detail in the proof of Proposition 6.8. Note that L'_X and L_X introduced in equations (7) resp. (8) of [19] correspond to $\mathrm{L}_{X,0}$ resp. $\mathrm{L}_{0,X}$ in the present setting.

For $K \subset\subset M$ given, let K_1 be some compact neighborhood of K. Since v is a continuous linear form on $\mathcal{T}^s_r(M) \otimes_{\mathcal{C}^\infty(M)} \Omega^n_{\mathrm{c}}(M)$, the seminorm estimate

for v on K_1 shows that there exist smooth vector fields $Y_j^{(i)}$ ($i = 1, \ldots, k$; $j = 1, \ldots, m_i$) and some $C > 0$ such that for any $w \in \mathcal{T}_r^s(M) \otimes \Omega_c^n(M)$ with $\mathrm{supp}(w) \subseteq K_1$ we have

$$|\langle v, w\rangle| \leq C \max_{i=1,\ldots,k} \|\mathrm{L}_{Y_1^{(i)}} \ldots \mathrm{L}_{Y_{m_i}^{(i)}} w\|_\infty.$$

For obtaining the moderateness estimates, it suffices to consider a single term $\|\mathrm{L}_{Y_1} \ldots \mathrm{L}_{Y_m} w\|_\infty$. If ε is sufficiently small, the arguments of v in the expression above for $\mathrm{L}_X(\iota_s^r(v)(\Phi(\varepsilon, p), p, A) \cdot \tilde{t}(p))$ both have support in K_1 with respect to (.). Furthermore, we obtain from the defining properties of smoothing kernels (cf. Definition 3.4) that the supremum over $p \in K$ of the first term is of order ε^{-n-m}. For the second, rewriting $\mathrm{L}_{X,0} = \mathrm{L}'_X$ as $\mathrm{L}_{X,X} - \mathrm{L}_{0,X}$ (corresponding to $(\mathrm{L}_X + \mathrm{L}'_X) - \mathrm{L}_X$ in [19]), we obtain an estimate of order ε^{-n-m-1}. Higher order Lie derivatives $\mathrm{L}_{X_1} \ldots \mathrm{L}_{X_l}$ can clearly be treated in the same way, yielding estimates of order ε^{-n-m-l}, as can derivatives with respect to A: As a formal calculation shows, the latter do not influence the order of ε since only the boundedness of A and $B_1, \ldots, B_j$ on K_1 is used. For interchanging the action of v with directional derivatives d_B in direction B with respect to A, we note that for p fixed, the map $\phi : A \mapsto (A_r^s(p, .)\, \tilde{t}(p)) \otimes \omega(.)$ is smooth from $\hat{\mathcal{B}}(M)$ into $\mathcal{T}_s^r(M) \otimes_{\mathcal{C}^\infty(M)} \Omega_c^n(M)$ with respect to the respective (LF)-topologies, as is, by definition, the linear map v. Hence by [26, 3.18], $\mathrm{d}_B\langle v, \phi(A)\rangle = \langle v, \mathrm{d}_B\phi(A)\rangle$, and the claim follows.

(ii) Since for $t \in \mathcal{T}_s^r(M)$, $\sigma_s^r(t)(\omega, p, A) = t(p)$ it is immediate that the $(\hat{\mathcal{E}}_s^r)_{\mathrm{m}}$-estimates hold for $\sigma_s^r(t)$ on any compact set, with $N = 0$.

(iii) Let $t \in \mathcal{T}_s^r(M)$, $K \subset\subset M$, $A \in \hat{\mathcal{B}}(M)$, $K \subseteq \mathrm{core}(A)$, and $\Phi \in \tilde{\mathcal{A}}_0(M)$. Then for any $\tilde{t} \in \mathcal{T}_r^s(M)$ and any $p \in K$

$$[(\sigma_s^r - \iota_s^r)(t)(\Phi(\varepsilon, p), A) \cdot \tilde{t}](p) = (t \cdot \tilde{t})(p) - \int_M t(q) A_r^s(p, q)\, \tilde{t}(p) \Phi(\varepsilon, p)(q)\, dq.$$

By Proposition 8.8 and Corollary 8.7 it suffices to show the negligibility estimates for this difference and its derivatives with respect to A. To this end we introduce the notation $f_p(q) := t(q) A_r^s(p, q)\tilde{t}(p)$. Then $f \in \mathcal{C}^\infty(M \times M)$ and the above expression reads

$$f_p(p) - \int_M f_p(q)\Phi(\varepsilon, p)(q)\, dq = \int_M (f_p(p) - f_p(q))\Phi(\varepsilon, p)(q)\, dq. \tag{8.2}$$

Lemma 3.6 now yields that (8.2) vanishes of order ε^{m+1}, uniformly for $p \in K$, for $\Phi \in \tilde{\mathcal{A}}_m(M)$. Next, we consider derivatives of $(\sigma_s^r - \iota_s^r)(t)$ with respect to A. Since $\sigma_s^r(t)$ does not depend on A we have to show that all A-derivatives of

$\iota_s^r(t)$ of order greater or equal one are negligible. To fix ideas we first consider the special case $r = 1$, $s = 0$. Then for A, $B \in \hat{\mathcal{B}}(M)$ and $\tilde{t} \in \mathcal{T}_1^0(M)$

$$(\mathrm{d}_3\iota_0^1(t))(\Phi(\varepsilon,p),p,A)(B)\cdot\tilde{t}(p) = \int_M t(q)\cdot B_0^1(p,q)\,\tilde{t}(p)\;\Phi(\varepsilon,p)(q)\,dq.$$

Now for $B \in \hat{\mathcal{B}}_{\mathrm{core}(A)}(M)$ and $p \in K \subset\subset \mathrm{core}(A)$, $B(p,p) = 0$, so again Lemma 3.6 gives the desired estimate. For general values of r and s, since $A \mapsto A_r^s$ is the composition of a multilinear map with the diagonal map, we obtain a sum of terms each of which has the form

$$\int_M f_p(q)\Phi(\varepsilon,p)(q)\,dq$$

with f smooth and $f_p(p) = 0$ for all p, so the claim follows by a third appeal to Lemma 3.6.

(iv) The (rather lengthy) direct proof would proceed along the lines of proof of Proposition 9.10 (the latter actually making a stronger statement than (iv) does). To minimize redundancy, we confine ourselves to noting that (iv) follows from Proposition 9.10, via Proposition 8.8 and Corollary 8.7 (with $m = 1$). □

Our next aim is to establish stability of $(\hat{\mathcal{E}}_s^r)_{\mathrm{m}}(M)$ and $\hat{\mathcal{N}}_s^r(M)$ under the Lie derivatives $\hat{\mathrm{L}}_X$ from Definition 6.7. Again we first consider the case $r = s = 0$:

Lemma 8.13. *Let $X \in \mathfrak{X}(M)$ and $u \in (\hat{\mathcal{E}}_0^0)_{\mathrm{m}}(M)$ resp. $u \in \hat{\mathcal{N}}_0^0(M)$. Then also $\hat{\mathrm{L}}_X u \in (\hat{\mathcal{E}}_0^0)_{\mathrm{m}}(M)$ resp. $\hat{\mathrm{L}}_X u \in \hat{\mathcal{N}}_0^0(M)$.*

Proof. The proof will be achieved by reduction (Lemma 8.6) to the setting of [19]. Recall from Definition 6.7 that

$$(\hat{\mathrm{L}}_X u)(\omega,p,A) = \mathrm{L}_X(u(\omega,A))(p) - \mathrm{d}_1 u(\omega,p,A)(\mathrm{L}_X\omega) - \mathrm{d}_3 u(\omega,p,A)(\mathrm{L}_X A).$$

Since $\mathrm{L}_X A \in \hat{\mathcal{B}}_{\mathrm{core}(A)}(M)$ for any $A \in \hat{\mathcal{B}}(M)$, the moderateness (resp. negligibility) estimates for $\hat{\mathrm{L}}_{0,0,X}u$, i.e., for $\mathrm{d}_3 u(\omega,p,A)(\mathrm{L}_X A)$ follow directly from the definitions. In order to show moderateness resp. negligibility of $\hat{\mathrm{L}}_{0,X,0}\,u+\hat{\mathrm{L}}_{X,0,0}\,u$, i.e. of $u_0 := (\omega,p,A) \mapsto \mathrm{L}_X(u(\omega,A))(p) - \mathrm{d}_1 u(\omega,p,A)(\mathrm{L}_X\omega)$ we employ Lemma 8.6. Fix $j \in \mathbb{N}_0$, $A \in \hat{\mathcal{B}}(M)$ and $B_1,\dots,B_j \in \hat{\mathcal{B}}_{\mathrm{core}(A)}(M)$. Then

$$\begin{aligned}&\mathrm{d}_3^j u_0(\omega,p,A)(B_1,\dots,B_j) = \\ &\mathrm{L}_X[\mathrm{d}_2^j u(\omega,A)(B_1,\dots,B_j)](p) - \mathrm{d}_1[\mathrm{d}_3^j u(\omega,p,A)(B_1,\dots,B_j)](\mathrm{L}_X\omega).\end{aligned} \tag{8.3}$$

In fact, we may interchange d_3 with d_1 due to symmetry of higher differentials ([26, 5.11]). Concerning d_3 and L_X we can either use Lemma 6.10 or note that due to $r = s = 0$ the term $\mathrm{L}_X(u(\omega, A))(p)$ can be written as $\mathrm{d}_2 u(\omega, p, A)(X)$ which again permits to resort to symmetry of higher differentials. (8.3) is precisely the Lie derivative in the sense of [19, Def. 3.8] of the map $(\omega, p) \mapsto \mathrm{d}_3^j u(\omega, p, A)(B_1, \ldots, B_j)$, hence is in $\hat{\mathcal{E}}_{\mathrm{m}}(\mathrm{core}(A))$ (resp. $\hat{\mathcal{N}}(\mathrm{core}(A))$) by [19, Th. 4.6]. Again by Lemma 8.6, the claim follows. □

Theorem 8.14. *$(\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M)$ and $\hat{\mathcal{N}}^r_s(M)$ are stable under Lie derivatives $\hat{\mathrm{L}}_X$ where $X \in \mathfrak{X}(M)$.*

Proof. It will suffice to treat the case of moderateness. Thus let $u \in (\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M)$ and $X \in \mathfrak{X}(M)$. Picking any $\tilde{t}$ from $\mathcal{T}^s_r(M)$, saturation (Proposition 8.8) yields $u \cdot \tilde{t} \in (\hat{\mathcal{E}}^0_0)_{\mathrm{m}}(M)$. By Lemma 8.13, also $\hat{\mathrm{L}}_X(u \cdot \tilde{t}) \in (\hat{\mathcal{E}}^0_0)_{\mathrm{m}}(M)$. However, $\hat{\mathrm{L}}_X(u \cdot \tilde{t}) = (\hat{\mathrm{L}}_X u) \cdot \tilde{t} + u \cdot (\hat{\mathrm{L}}_X \tilde{t})$ due to Proposition 6.9. The second term being a member of $(\hat{\mathcal{E}}^0_0)_{\mathrm{m}}(M)$, again by Proposition 8.8, we infer that $(\hat{\mathrm{L}}_X u) \cdot \tilde{t}$ is moderate. Since $\tilde{t}$ was arbitrary, a third appeal to Proposition 8.8 establishes the moderateness of $\hat{\mathrm{L}}_X u$. □

Thus we finally arrive at

Definition 8.15. The space of generalized (r, s)-tensor fields is defined as

$$\hat{\mathcal{G}}^r_s(M) := (\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M) / \hat{\mathcal{N}}^r_s(M).$$

$\hat{\mathcal{G}}^r_s(M)$ is both a $\mathcal{C}^\infty(M)$- and a $\hat{\mathcal{G}}^0_0(M)$-module. For $u \in (\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M)$ we denote by $[u]$ its equivalence class in $\hat{\mathcal{G}}^r_s(M)$. From Theorem 8.12 it follows that ι^r_s and σ^r_s induce maps from $\mathcal{D}'^r_s(M)$ resp. $\mathcal{T}^r_s(M)$ into $\hat{\mathcal{G}}^r_s(M)$. These maps will be denoted by the same letters. We collect the main properties of $\hat{\mathcal{G}}^r_s(M)$ in the following result.

Theorem 8.16. *The map*

$$\iota^r_s : \mathcal{D}'^r_s(M) \to \hat{\mathcal{G}}^r_s(M)$$

is a linear embedding whose restriction to $\mathcal{T}^r_s(M)$ coincides with

$$\sigma^r_s : \mathcal{T}^r_s(M) \to \hat{\mathcal{G}}^r_s(M)\,.$$

For any smooth vector field X on M, the Lie derivative

$$\begin{aligned} \hat{\mathrm{L}}_X : \hat{\mathcal{G}}^r_s(M) &\to \hat{\mathcal{G}}^r_s(M) \\ \hat{\mathrm{L}}_X([u]) &:= [\hat{\mathrm{L}}_X u] \end{aligned}$$

is a well-defined operation commuting with the embedding, i.e., for any $v \in \mathcal{D}'^r_s(M)$, $\iota^r_s(\mathrm{L}_X v) = \hat{\mathrm{L}}_X \iota^r_s(v)$.

Proof. All the claimed properties of ι^r_s and σ^r_s follow from Theorem 8.12. $\hat{\mathrm{L}}_X$ is well-defined by Theorem 8.14. Finally, that Lie derivatives commute with the embedding was already established in Proposition 6.8. □

Summing up, we obtain the following commutative diagram:

$$
\begin{tikzcd}
\mathcal{T}^r_s(M) \arrow[rrr, "\mathrm{L}_X"] \arrow[dr, "\rho^r_s"] \arrow[dd, "\sigma^r_s"'] & & & \mathcal{T}^r_s(M) \arrow[dl, "\rho^r_s"'] \arrow[dd, "\sigma^r_s"] \\
& \mathcal{D}'^r_s(M) \arrow[r, "\mathrm{L}_X"] \arrow[dl, "\iota^r_s"'] & \mathcal{D}'^r_s(M) \arrow[dr, "\iota^r_s"] & \\
\hat{\mathcal{G}}^r_s(M) \arrow[rrr, "\hat{\mathrm{L}}_X"] & & & \hat{\mathcal{G}}^r_s(M)
\end{tikzcd}
$$

As was highlighted in Section 4, the properties included in this diagram are optimal in light of Schwartz' impossibility result.

To extend these results to the universal tensor algebra over M, we first note that

$$
\begin{aligned}
(\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M) \otimes (\hat{\mathcal{E}}^{r'}_{s'})_{\mathrm{m}}(M) &\subseteq (\hat{\mathcal{E}}^{r+r'}_{s+s'})_{\mathrm{m}}(M) \\
(\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M) \otimes \hat{\mathcal{N}}^{r'}_{s'}(M) &\subseteq \hat{\mathcal{N}}^{r+r'}_{s+s'}(M).
\end{aligned}
$$

Thus we obtain the algebra $\mathcal{T}_{\hat{\mathcal{E}}_{\mathrm{m}}}(M) := \bigoplus_{r,s} (\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M)$ containing the ideal $\mathcal{T}_{\hat{\mathcal{N}}}(M) := \bigoplus_{r,s} \hat{\mathcal{N}}^r_s(M)$.

Definition 8.17. The universal algebra of generalized tensor fields is defined as

$$
\mathcal{T}_{\hat{\mathcal{G}}}(M) := \mathcal{T}_{\hat{\mathcal{E}}_{\mathrm{m}}}(M)/\mathcal{T}_{\hat{\mathcal{N}}}(M) \cong \bigoplus_{r,s} (\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M)/\hat{\mathcal{N}}^r_s(M) = \bigoplus_{r,s} \hat{\mathcal{G}}^r_s(M).
$$

The operations of tensor product, contraction and Lie derivative with respect to smooth vector fields naturally extend to $\mathcal{T}_{\hat{\mathcal{G}}}(M)$ and we have, by Proposition 6.9,

$$
\hat{\mathrm{L}}_X(u_1 \otimes u_2) = (\hat{\mathrm{L}}_X u_1) \otimes u_2 + u_1 \otimes (\hat{\mathrm{L}}_X u_2).
$$

Furthermore, the embeddings ι^r_s and σ^r_s extend to $\mathcal{T}_{\mathcal{D}'}(M) := \bigoplus_{r,s} \mathcal{D}'^r_s(M)$ resp. $\mathcal{T}(M) := \bigoplus_{r,s} \mathcal{T}^r_s(M)$. We will denote the respective maps by ι resp. σ. From Theorem 8.16 we obtain:

Corollary 8.18. *The mapping*

$$\iota : \mathcal{T}_{\mathcal{D}'}(M) \to \mathcal{T}_{\hat{\mathcal{G}}}(M)$$

is a linear embedding whose restriction to $\mathcal{T}(M)$ coincides with the algebra homomorphism

$$\sigma : \mathcal{T}(M) \to \mathcal{T}_{\hat{\mathcal{G}}}(M),$$

thereby rendering $\mathcal{T}(M)$ a subalgebra of $\mathcal{T}_{\hat{\mathcal{G}}}(M)$. For any smooth vector field X on M, the Lie derivatives $\hat{\mathrm{L}}_X : \mathcal{T}_{\hat{\mathcal{G}}}(M) \to \mathcal{T}_{\hat{\mathcal{G}}}(M)$ resp. $\mathrm{L}_X : \mathcal{T}(M) \to \mathcal{T}(M)$ intertwine with the embedding ι.

To conclude this section, we give the following characterization of $\hat{\mathcal{G}}^r_s(M)$ as a $\mathcal{C}^\infty(M)$-module.

Theorem 8.19. *The following chain of $\mathcal{C}^\infty(M)$-module isomorphisms holds:*

$$\hat{\mathcal{G}}^r_s(M) \cong \hat{\mathcal{G}}^0_0(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^r_s(M) \cong \mathrm{L}_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \hat{\mathcal{G}}^0_0(M)).$$

Proof. The $\mathcal{C}^\infty(M)$-module $\mathcal{T}^r_s(M)$ is projective and finitely generated (cf. [14, 2.23], applied to each connected (hence second countable) component of M). Thus by [2, Ch. II, §4, 2], it follows that

$$\begin{aligned}\mathrm{L}_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \hat{\mathcal{G}}^0_0(M)) &\cong \hat{\mathcal{G}}^0_0(M) \otimes_{\mathcal{C}^\infty(M)} \mathrm{L}_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{C}^\infty(M)) \\ &= \hat{\mathcal{G}}^0_0(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^r_s(M).\end{aligned}$$

We establish the theorem by showing $\hat{\mathcal{G}}^r_s(M) \cong \mathrm{L}_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \hat{\mathcal{G}}^0_0(M))$. By the exponential law in [26, 27.17], it is immediate from Lemma 7.2 (4) (or (3)) that

$$\begin{aligned}\hat{\mathcal{E}}^r_s(M) &\cong \mathrm{L}^b_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{C}^\infty(\hat{\mathcal{A}}_0(M) \times \hat{\mathcal{B}}(M)\,, \mathcal{C}^\infty(M))) \\ &= \mathrm{L}^b_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M)\,, \hat{\mathcal{E}}^0_0(M)).\end{aligned}$$

holds. Here, the boundedness assumption in the last term can be formally dropped, i.e., $\mathrm{L}^b_{\mathcal{C}^\infty(M)}$ can safely be replaced by $\mathrm{L}_{\mathcal{C}^\infty(M)}$: Since all the spaces involved are convenient, a $\mathcal{C}^\infty(M)$-linear map $F : \mathcal{T}^s_r(M) \to \hat{\mathcal{E}}^0_0(M)$ is bounded if and only if for all $\omega \in \hat{\mathcal{A}}_0(M)$, $A \in \hat{\mathcal{B}}(M)$, the maps $F_{\omega,A} : \tilde{t} \mapsto F(\tilde{t})(\omega, A)$ (for $\tilde{t} \in \mathcal{T}^s_r(M)$) are bounded, due to the uniform boundedness principle [26, 5.26]. Being a member of $\mathrm{L}_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M)\,, \mathcal{C}^\infty(M))$, however, the map $F_{\omega,A}$ is of the form $\tilde{t} \mapsto t \cdot \tilde{t}$ for some $t \in \mathcal{T}^r_s(M)$ and thus even continuous with respect to the Fréchet topologies. Using saturation (Proposition 8.8), it is straightforward to check that $\hat{\mathcal{E}}^r_s(M) \cong \mathrm{L}_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M)\,, \hat{\mathcal{E}}^0_0(M))$ induces an isomorphism from $\hat{\mathcal{G}}^r_s(M)$ onto $\mathrm{L}_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M)\,, \hat{\mathcal{G}}^0_0(M))$, thereby finishing the proof. □

9 Association

In all versions of Colombeau's construction the Schwartz impossibility result is circumvented by introducing a very narrow concept of equality, more precisely, by introducing a very strict equivalence relation on the space of moderate elements. In particular, this equivalence is finer than distributional equality. Nevertheless, raising the latter to the level of the algebra by introducing an equivalence relation called association one can take advantage of using both notions of equality in the so-called "coupled calculus". For example, tensor products of continuous or $\mathcal{C}^k$-fields are not preserved in the algebra $\mathcal{T}_{\hat{\mathcal{G}}}$ in the sense that the embedding is not a homomorphism with respect to the tensor product. It will, however, turn out to to be a homomorphism at the level of association.

In many situations of practical relevance, elements of the algebra are associated to distributions. This feature has the advantage that often one may use the mathematical power of the differential algebra to perform the calculations but then invoke the notion of association to give a physical interpretation to the result obtained. This is especially useful when it comes to modelling source terms in nonlinear partial differential equations and, consequently, one often wants to consider such equations in the sense of association rather than equality (cf., e.g., [7, 34]). One of the applications we have in mind is Einstein's equations where we seek generalized metrics which have an Einstein tensor associated to a distributional energy-momentum tensor representing, e.g., a cosmic string or a shell of matter (cf. [3, 40, 39] and the references therein).

In this section we introduce an appropriate concept of association for generalized tensor fields.

Note that as an exception to our standard notation, in this section we will use capitals for generalized scalar and tensor fields. This will permit us to distinguish notationally between elements u_1, u_2 etc. of $(\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M)$ and their respective classes $U_1 = [u_1]$, $U_2 = [u_2]$ etc. in $\hat{\mathcal{G}}^r_s(M)$. We start by briefly considering the scalar case (touched upon in [19]).

Definition 9.1. We say that a generalized scalar field $F = [f] \in \hat{\mathcal{G}}(M)$ is associated with 0 (denoted $F \approx 0$), if for some (and hence any) representative $f \in \hat{\mathcal{E}}_{\mathrm{m}}(M)$ of F and for each $\omega \in \Omega^n_{\mathrm{c}}(M)$ there exists some $m > 0$ such that $\forall \Phi \in \tilde{\mathcal{A}}_m(M)$

$$\lim_{\varepsilon \to 0} \int_M f(\Phi(\varepsilon, p), p)\omega(p) = 0\,.$$

We say that two generalized functions F, G are associated and write $F \approx G$ if $F - G \approx 0$.

At the level of association we regain the usual results for multiplication of distributions ([19, Prop. 6.2]).

Proposition 9.2.

(i) If $f \in \mathcal{C}^\infty(M)$ and $v \in \mathcal{D}'(M)$ then

$$\iota(f)\iota(v) \approx \iota(fv).$$

(ii) If $f, g \in \mathcal{C}(M)$ then

$$\iota(f)\iota(g) \approx \iota(fg).$$

It is also useful to introduce the concept of associated distribution or "distributional shadow" of a generalized function.

Definition 9.3. We say that $F \in \hat{\mathcal{G}}(M)$ admits $v \in \mathcal{D}'(M)$ as an associated distribution if $F \approx \iota(v)$.

The notion of associated distribution can be expressed through the following concept of convergence.

Definition 9.4. We say that $F \in \hat{\mathcal{G}}(M)$ converges weakly to $v \in \mathcal{D}'(M)$, and write $F \overset{\mathcal{D}'(M)}{\rightarrow} v$ if for some (hence any) representative $f \in \hat{\mathcal{E}}_{\mathrm{m}}(M)$ of F and for each $\omega \in \Omega^n_{\mathrm{c}}(M)$ there exists some $m > 0$ so that $\forall \Phi \in \tilde{\mathcal{A}}_m(M)$

$$\lim_{\varepsilon \to 0} \int_M f(\Phi(\varepsilon, p), p)\omega(p) = \langle v, \omega \rangle.$$

In fact the following proposition states that weak convergence to v is equivalent to having v as an associated distribution.

Proposition 9.5. *An element $F = [f]$ of $\hat{\mathcal{G}}(M)$ possesses $v \in \mathcal{D}'(M)$ as an associated distribution if and only if $F \overset{\mathcal{D}'(M)}{\rightarrow} v$.*

The proof of Proposition 9.5 is a slimmed-down version of that of Proposition 9.10, compare Corollary 9.11. Note that not all generalized functions have a distributional shadow. However, if $v \in \mathcal{D}'(M)$ and $\iota(v) \approx 0$ then $v = 0$, so that provided the distributional shadow exists it is unique.

We now extend this circle of ideas to the tensor case. We start by defining association for generalized tensor fields.

Definition 9.6. A generalized tensor field $U = [u] \in \hat{\mathcal{G}}^r_s(M)$ is called associated with 0, $U \approx 0$, if for one (hence any) representative u, we have:

$$\forall \omega \in \Omega^n_{\mathrm{c}}(M)\ \forall A \in \hat{\mathcal{B}}(M)\ \forall \tilde{t} \in \mathcal{T}^s_r(M)\ \exists m > 0\ \forall \Phi \in \tilde{\mathcal{A}}_m(M):$$

$$\lim_{\varepsilon \to 0} \int u(\Phi(\varepsilon, p), p, A)\, \tilde{t}(p)\, \omega(p) = 0.$$

U_1, $U_2 \in \hat{\mathcal{G}}^r_s(M)$ are called associated, $U_1 \approx U_2$, if $U_1 - U_2 \approx 0$.

Definition 9.7. A generalized tensor field $U \in \hat{\mathcal{G}}^r_s(M)$ is said to admit $v \in \mathcal{D}'^r_s(M)$ as an associated distribution and v is called the distributional shadow of U, if $U \approx \iota^r_s(v)$.

Employing the localization techniques from the proof of Proposition 8.10 we obtain:

Lemma 9.8. *The following statements are equivalent for any $U = [u] \in \hat{\mathcal{G}}^r_s(M)$:*

(i) $U \approx 0$ *in* $\hat{\mathcal{G}}^r_s(M)$.

(ii) $\forall \tilde{t} \in \mathcal{T}^s_r(M)$*:* $U \cdot \tilde{t} \approx 0$ *in* $\hat{\mathcal{G}}^0_0(M)$.

(iii) $\forall W \subseteq M$ *open:* $\forall \omega \in \Omega^n_c(W)\ \forall A \in \hat{\mathcal{B}}(M)\ \forall \tilde{t} \in \mathcal{T}^s_r(W)\ \exists m > 0 :\ \forall \Phi \in \tilde{\mathcal{A}}_m(W) :$
$$\lim_{\varepsilon \to 0} \int u(\Phi(\varepsilon, p), p, A)\, \tilde{t}(p)\, \omega(p) = 0.$$

Definition 9.9. We say that a generalized tensor field $U \in \hat{\mathcal{G}}^r_s(M)$ converges weakly to $v \in \mathcal{D}'^r_s(M)$, and write $U \overset{\mathcal{D}'(M)}{\to} v$ if for some (hence any) representative $u \in (\hat{\mathcal{E}}^r_s)_{\mathrm{m}}(M)$ of U we have

$$\begin{gathered}\forall \omega \in \Omega^n_c(M)\ \forall A \in \hat{\mathcal{B}}(M)\ \forall \tilde{t} \in \mathcal{T}^s_r(M)\ \exists m > 0\ \forall \Phi \in \tilde{\mathcal{A}}_m(M) : \\ \lim_{\varepsilon \to 0} \textstyle\int u(\Phi(\varepsilon, p), p, A)\, \tilde{t}(p)\, \omega(p) = \langle v, \tilde{t} \otimes \omega \rangle.\end{gathered}$$

In the proof of the following result we will make use of a refined version of (the technical core of) Theorem 16.5 of [17]: For W an open subset of $\mathbb{R}^n$ let $c : D(\subseteq I \times W) \to \mathbb{R}$ denote a smooth function in the sense of (5) and (6) of [19] (with the range space $\mathcal{A}_0(\mathbb{R}^n)$ resp. $\mathcal{D}(\mathbb{R}^n)$ replaced by $\mathbb{R}$). Moreover, let K, L be compact subsets of W such that $K \subset\subset L \subset\subset W$ and $(0, \varepsilon_0) \times L$ is contained in the interior of D. Finally, let $q > 0$, $\eta > 0$. If $\sup_{x \in L} |c(\varepsilon, x)| = O(\varepsilon^q)$ then it follows that $\sup_{x \in K} |\partial^\beta c(\varepsilon, x)| = O(\varepsilon^{q-\eta})$ for every $\beta \in \mathbb{N}^n_0$. This can be established along the lines of the proof of Theorem 16.5 of [17].

Proposition 9.10. *Let $v \in \mathcal{D}'^r_s(M)$. Then*

$$\begin{gathered}\forall \omega \in \Omega^n_c(M)\ \forall A \in \hat{\mathcal{B}}(M)\ \forall \tilde{t} \in \mathcal{T}^s_r(M)\ \forall \Phi \in \tilde{\mathcal{A}}_0(M) : \\ \lim_{\varepsilon \to 0} \textstyle\int \iota^r_s(v)(\Phi(\varepsilon, p), p, A)\, \tilde{t}(p)\, \omega(p) = \langle v, \tilde{t} \otimes \omega \rangle.\end{gathered} \tag{9.1}$$

In particular, $\iota^r_s(v)$ satisfies the conditions of Definition 9.9 with $m = 0$, so

$$\iota^r_s(v) \overset{\mathcal{D}'}{\to} v.$$

Proof. Let ω, A, $\tilde{t}$ and Φ be given. Since both sides of (9.1) are linear in ω we may assume that $\operatorname{supp}\omega \subset\subset W$ where (W,ψ) is a chart on M. Let us fix compact subsets L', L of W with $\operatorname{supp}\omega \subset\subset L' \subset\subset L \subset\subset W$. We may suppose without loss of generality that the images of W, L', L under ψ are balls in $\mathbb{R}^n$. By the defining properties of a smoothing kernel there exists $\varepsilon_0 > 0$ such that for $\varepsilon \le \varepsilon_0$ and $p \in L'$ we have $\operatorname{supp}\Phi(\varepsilon, p) \subset\subset L$. Thus we may further assume without loss of generality that also $\operatorname{supp} v \subset\subset W$. Passing to coordinates we may therefore suppose that W is an (open) ball in $\mathbb{R}^n$, with $\tilde{t}$ and A defined on W resp. $W \times W$ and ω, v compactly supported in W. Finally, by Lemma 4.2 of [19], the place of Φ is taken by $\varepsilon^{-n}\phi(\varepsilon,x)(\frac{y-x}{\varepsilon})$ where $\phi \in \mathcal{C}^\infty_{b,w}(I \times W, \mathcal{A}_0(\mathbb{R}^n))$ is defined by

$$\phi(\varepsilon,x)(y)\, d^n y := \varepsilon^n((\psi^{-1})^*\Phi(\varepsilon,\psi^{-1}(x)))(\varepsilon y + x).$$

Writing x, y for p, q and $\varphi\, d^n x$ for ω, we obtain that for $\varepsilon \le \varepsilon_0$, the expression $\int \iota^r_s(v)(\Phi(\varepsilon,p),p,A)\tilde{t}(p)\omega(p) - \langle v, \tilde{t}\otimes\omega\rangle$ locally takes the form

$$\begin{aligned}
&\int \Big\langle v(y), A^s_r(x,y)\tilde{t}(x)\varepsilon^{-n}\phi(\varepsilon,x)\left(\tfrac{y-x}{\varepsilon}\right)\Big\rangle\varphi(x)\, d^n x - \langle v(y), \tilde{t}(y)\varphi(y)\rangle \\
&\quad = \Big\langle v(y), \int \big(A^s_r(y-\varepsilon z, y)\tilde{t}(y-\varepsilon z)\varphi(y-\varepsilon z)\phi(\varepsilon, y-\varepsilon z)(z) \\
&\qquad\qquad\qquad - A^s_r(y,y)\tilde{t}(y)\varphi(y)\phi(\varepsilon,y)(z)\big)\, d^n z\Big\rangle.
\end{aligned}$$

Since $\phi : (0,\varepsilon_0)\times(L')^\circ \to \mathcal{A}_0(\mathbb{R}^n)$ is smooth with $\operatorname{supp}\phi(\varepsilon,x)(\frac{.-x}{\varepsilon}) \subseteq L$ for all ε, x and the evaluation map $\mathrm{ev} : \mathcal{A}_0(\mathbb{R}^n)\times\mathbb{R}^n \to \mathbb{R}$ is smooth by [26, 3.13 (i)], the expression $J(x,y) := A^s_r(x,y)\tilde{t}(x)\varepsilon^{-n}\phi(\varepsilon,x)(\frac{y-x}{\varepsilon})\varphi(x)$ represents a member of $\mathcal{D}(\mathbb{R}^n\times\mathbb{R}^n)^{n^{r+s}}$ (for $\varepsilon < \varepsilon_0$) with $\operatorname{supp} J \subseteq \operatorname{supp}\varphi\times L$. Therefore, the combined action of v and integration with respect to x can be viewed as the action of the distribution $\mathbb{1}(x)\otimes v(y)$ on J, allowing to interchange v with the integral. Since $\phi \in \mathcal{A}^{\triangle}_{m,w}(W)$ by Lemma 4.2(A) of [19], we have $\sup_{\xi\in L}|c(\varepsilon,\xi)| = O(\varepsilon^{m+1-|\alpha|})$ for $c(\varepsilon,x) := \int_{\mathbb{R}^n}\phi(\varepsilon,x)(z)z^\alpha\, \mathrm{d}^n z$ and $1 \le |\alpha| \le m$. From the analogue of Theorem 16.5 of [17] discussed above we infer, for every $\beta \in \mathbb{N}^n_0$,

$$\sup_{\xi\in L'}|\partial^\beta c(\varepsilon,\xi)| = \sup_{\xi\in L'}\left|\int_{\mathbb{R}^n}\partial^\beta\phi(\varepsilon,\xi)(z)z^\alpha\, \mathrm{d}^n z\right| = O(\varepsilon^{m+1-|\alpha|-\eta}).$$

Now, applying Taylor expansion of order m to every tensor component of

$$\begin{aligned}
\psi_\varepsilon(z,y) := A^s_r(y-\varepsilon z, y)\,\tilde{t}(y-\varepsilon z)\varphi(y-\varepsilon z)\phi(\varepsilon, y-\varepsilon z)(z)& \\
- A^s_r(y,y)\tilde{t}(y)\varphi(y)\phi(\varepsilon,y)(z)&
\end{aligned}$$

and integrating with respect to z we obtain estimates of order $\varepsilon^{m+1-\eta}$ for the terms of the Taylor polynomials and of order ε^{m+1} for the respective remainder terms, uniformly for $y \in L'$. For $m = 0$, the Taylor expansions consist of the remainder terms solely, allowing overall estimates even by ε^{m+1}. Moreover, $\psi_\varepsilon(z, y)$ vanishes for $y \notin L'$. On the basis of analogous asymptotics for $\int \partial_y^\beta \psi_\varepsilon(z, y)\, \mathrm{d}^n z$ it follows that $\varepsilon^{-(m+1-\eta)} \cdot \int \psi_\varepsilon(z, y)\, \mathrm{d}^n z$ is bounded in $\mathcal{D}(\mathbb{R}^n)^{n^{r+s}}$. Altogether, we obtain $\langle v(.), \int \psi_\varepsilon(z, .)\, \mathrm{d}^n z\rangle$ being of order $\varepsilon^{m+1-\eta}$ resp. ε^{m+1} (for $m = 0$), thereby establishing our claim. $\square$

Corollary 9.11. *An element U of $\hat{\mathcal{G}}^r_s(M)$ possesses $v \in \mathcal{D}'^r_s(M)$ as an associated distribution if and only if $U \overset{\mathcal{D}'}{\to} v$.*

It follows from Corollary 9.11 and Proposition 9.10 that the distributional shadow of a generalized tensor field is unique (if it exists).

For continuous tensor fields we obtain stronger (locally uniform) convergence properties:

Proposition 9.12. *Let t be a continuous (r, s)-tensor field on M. Then*

$$\forall K \subset\subset M\ \forall A \in \hat{\mathcal{B}}(M)\ \forall \tilde{t} \in \mathcal{T}^s_r(M)\ \forall \Phi \in \tilde{\mathcal{A}}_0(M):$$
$$\lim_{\varepsilon \to 0} \sup_{p \in K} \left| \iota^r_s(t)(\Phi(\varepsilon, p), p, A)\tilde{t}(p) - t(p) \cdot \tilde{t}(p) \right| = 0.$$

As a consequence of this result and the proof of Proposition 9.10 we obtain compatibility of the embedding ι^r_s with the standard products on $\mathcal{C} \times \mathcal{C}$ and $\mathcal{C}^\infty \times \mathcal{D}'$ in the sense of association:

Corollary 9.13.

(i) Let $t \in \mathcal{T}^{r_1}_{s_1}$, $v \in \mathcal{D}'^{r_2}_{s_2}$. Then $\iota^{r_1}_{s_1}(t) \otimes \iota^{r_2}_{s_2}(v) \approx \iota^{r_1+s_1}_{r_2+s_2}(t \otimes v)$.

(ii) Let t_1, t_2 be continuous tensor fields of order (r_1, s_1) resp. (r_2, s_2). Then $\iota^{r_1}_{s_1}(t_1) \otimes \iota^{r_2}_{s_2}(t_2) \approx \iota^{r_1+s_1}_{r_2+s_2}(t_1 \otimes t_2)$.

Remark 9.14. Guided by the notion of convergence given in Proposition 9.12, we may introduce the concept of $\mathcal{C}^0$-association: $U = [u] \in \hat{\mathcal{G}}^r_s(M)$ is called $\mathcal{C}^0$-associated with 0, $U \approx_0 0$, if for one (hence any) representative u, we have:

$$\forall K \subset\subset M\ \forall A \in \hat{\mathcal{B}}(M)\ \forall \tilde{t} \in \mathcal{T}^s_r(M)\ \exists m > 0\ \forall \Phi \in \tilde{\mathcal{A}}_m(M):$$
$$\lim_{\varepsilon \to 0} \sup_{p \in K} \left| u(\Phi(\varepsilon, p), p, A)\tilde{t}(p) \right| = 0.$$

$U_1, U_2 \in \hat{\mathcal{G}}^r_s(M)$ are called $\mathcal{C}^0$-associated, $U_1 \approx_0 U_2$, if $U_1 - U_2 \approx_0 0$. Moreover, for t a continuous (r,s)-tensor field we write $U \approx_0 t$ if $U \approx_0 \iota^r_s(t)$.

Analogously we may introduce the concept of $\mathcal{C}^k$-association by considering $\mathcal{C}^k$-convergence instead of $\mathcal{C}^0$-convergence in the above definition ($k \in \mathbb{N}_0$). With this notion, Corollary 9.13 (ii) can be strengthened: if t_1, t_2 are $\mathcal{C}^k$-tensor fields then $\iota^{r_1}_{s_1}(t_1) \otimes \iota^{r_2}_{s_2}(t_2) \approx_k \iota^{r_1+s_1}_{r_2+s_2}(t_1 \otimes t_2)$. In fact, for any $\mathcal{C}^k$-(r,s)-tensor field t, $\iota^r_s(t) \approx_k t$.

Appendix A Transport operators and two-point tensors

In this appendix we collect the main definitions, notations and properties of transport operators resp. two-point tensors.

Let M, N be smooth paracompact Hausdorff manifolds of (finite) dimensions n and m, respectively. We consider the vector bundle

$$\mathrm{TO}(M,N) := \mathrm{L}_{M\times N}(\mathrm{T}M, \mathrm{T}N) := \bigcup_{(p,q)\in M\times N} \{(p,q)\} \times \mathrm{L}(\mathrm{T}_pM, \mathrm{T}_qN)$$

of transport operators on $M \times N$. For charts $(U, \varphi = (x^1, \dots, x^n))$, $(V, \psi = (y^1, \dots, y^m))$ of M resp. N, a typical vector bundle chart (or vb-chart, for short) of $\mathrm{TO}(M,N)$ is given by

$$\begin{aligned} \Psi_{\varphi\psi} : \bigcup_{(p,q)\in U\times V} \{(p,q)\} \times \mathrm{L}(\mathrm{T}_pM, \mathrm{T}_qN) &\to \varphi(U) \times \psi(V) \times \mathbb{R}^{mn} \\ ((p,q), A) &\mapsto ((\varphi(p), \psi(q)), (dy^i(A\partial_{x_j}))_{i,j})\,. \end{aligned}$$

Setting $\varphi_{21} := \varphi_2 \circ \varphi_1^{-1}$ and analogously for ψ, the transition functions for $\mathrm{TO}(M,N)$ are given by

$$\Psi_{\varphi_2\psi_2} \circ \Psi^{-1}_{\varphi_1\psi_1}((x,y),a) = ((\varphi_{21}(x), \psi_{21}(y)), D\psi_{21}(y) \cdot a \cdot D\varphi_{21}(x)^{-1})\,.$$

Elements of $\Gamma(\mathrm{TO}(M,N))$, i.e., smooth sections of $\mathrm{TO}(M,N)$, are also called transport operators.

Alternatively, transport operators can be viewed as sections of a suitable pullback bundle. To fix notations (following [11]), let $E \xrightarrow{\pi} B$ be a vector bundle, B' a manifold and $f : B' \to B$ a smooth map. We denote by $E' = f^*(E)$ the pullback bundle of E under f. The total space of $f^*(E)$ is the closed submanifold $B' \times_B E := \{(b', e) \in B' \times E \mid f(b') = \pi(e)\}$ of

$B' \times E$, and the projection is $\pi' = pr_1|_{B' \times_B E}$, i.e., we have the following diagram, where $f' := \mathrm{pr}_2|_{B' \times_B E}$:

$$\begin{array}{ccc} E' & \xrightarrow{f'} & E \\ {\scriptstyle \pi'}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\ B' & \xrightarrow[f]{} & B \end{array}$$

If $(\pi^{-1}(U), \Psi)$ is a vb-chart for E with $\Psi = (e_b \mapsto (\Psi^{(1)}(b), \Psi^{(2)}(e_b)))$ and if (V, φ) is a chart in B' such that $V \cap f^{-1}(U) \neq \emptyset$, then

$$\begin{aligned} \varphi \times_B \Psi : {\pi'}^{-1}(V \cap f^{-1}(U)) &\to \varphi(V) \times \mathbb{R}^N \\ (b', e) &\mapsto (\varphi(b'), \Psi^{(2)}(e)) \end{aligned}$$

is a typical vb-chart for $f^*(E)$. To obtain the explicit form of the transition functions of $f^*(E)$, let $\varphi_{21} := \varphi_2 \circ \varphi_1^{-1}$ be a change of charts in B' and $\Psi_2 \circ \Psi_1^{-1}(x, \xi) = (\Psi_{21}^{(1)}(x), \Psi_{21}^{(2)}(x) \cdot \xi)$ a change of vb-charts in E. Then the corresponding change of vb-charts in $f^*(E)$ is given by

$$(\varphi_2 \times_B \Psi_2) \circ (\varphi_1 \times_B \Psi_1)^{-1}(x', \eta) = (\varphi_{21}(x'), \Psi_{21}^{(2)}(\Psi_1^{(1)} \circ f \circ \varphi_1^{-1}(x')) \cdot \eta).$$

For the particular case where f is of the form $\mathrm{pr}_2 : M \times N \to N$ for manifolds M, N and a vector bundle $E \xrightarrow{\pi} N$, there is a simplified way of representing $\mathrm{pr}_2^* E$ as a vector bundle over $M \times N$ (a similar statement being true for $f = \mathrm{pr}_1$ and a vector bundle $E \xrightarrow{\pi} M$) which we will use freely wherever convenient: Namely, $\mathrm{pr}_2^* E$ can be realized in this case as the (full) product manifold $M \times E$ (rather than as $(M \times N) \times_N E$), with projection $\mathrm{id}_M \times \pi : (p, v) \mapsto (p, \pi(v))$.

Now for M, N as above and pr_1 and pr_2 the projection maps of $M \times N$ onto M resp. N, we apply the above constructions to obtain the bundle

$$\mathrm{TP}(M, N) := (\mathrm{pr}_1^*(\mathrm{T}^* M) \otimes \mathrm{pr}_2^*(\mathrm{T}N), M \times N, \pi_\otimes)$$

of *two-point tensors* on $M \times N$.

Sections of $\mathrm{TP}(M, N)$ will also be called two-point tensors. Note that any element of $\Gamma(\mathrm{TP}(M, N))$ is a finite sum of sections of the form

$$(p, q) \mapsto f(p, q)\, \eta(p) \otimes \xi(q) \tag{A.1}$$

where $\eta \in \Omega^1(M)$, $\xi \in \mathfrak{X}(N)$ and $f \in \mathcal{C}^\infty(M \times N)$. We will call two-point tensors of this form *generic*.

We will use the following notation for the obvious connection between transport operators and two-point tensors: For V, W finite dimensional vector spaces we have the canonical isomorphism

$$\begin{aligned} \bullet : V^* \otimes W &\rightarrow \mathrm{L}(V, W) \\ (\alpha \otimes w)_\bullet(v) &:= \langle \alpha, v \rangle w \end{aligned}$$

which induces a strong vb-isomorphism (in the sense of [14, ch. II, §1])

$$\bullet : \mathrm{TP}(M, N) \rightarrow \mathrm{TO}(M, N)\,. \tag{A.2}$$

For a two-point tensor $\Upsilon \in \Gamma(\mathrm{TP}(M, N))$ we denote by $\Upsilon_\bullet$ the corresponding transport operator in $\Gamma(\mathrm{TO}(M, N))$. Viewing transport operators as two-point tensors (as was done in [42]) has some advantages when doing explicit calculations (cf., e.g., the formula for the Lie derivative (A.7) below). Moreover, we note that $\mathrm{TP}(M, N)$ is canonically isomorphic to the first jet bundle $J^1(M, N)$ (cf. [25, 12.9]). $\mathrm{TP}(M, N)$ also appears as the particular case $\mathrm{T}^*M \boxtimes \mathrm{T}N$ of the so-called *external tensor product* $E \boxtimes F$ of vector bundles $E \rightarrow M$, $F \rightarrow N$ in [14, Ch. II, Problem 4].

Given diffeomorphisms $\mu : M_1 \rightarrow M_2$ and $\nu : N_1 \rightarrow N_2$, we have a natural pullback action

$$(\mu, \nu)^* : \Gamma(\mathrm{TO}(M_2, N_2)) \rightarrow \Gamma(\mathrm{TO}(M_1, N_1))\,,$$

given (for a transport operator $A \in \Gamma(\mathrm{TO}(M_2, N_2))$) by

$$((\mu, \nu)^* A)(p, q) = (\mathrm{T}_q\nu)^{-1} \circ A(\mu(p), \nu(q)) \circ \mathrm{T}_p\mu\,. \tag{A.3}$$

Similarly, we obtain a natural pullback action

$$(\mu, \nu)^* : \Gamma(\mathrm{TP}(M_2, N_2)) \rightarrow \Gamma(\mathrm{TP}(M_1, N_1)),$$

defined on generic two-point tensors by

$$\begin{aligned} ((\mu, \nu)^*(f\eta \otimes \xi))(p, q) :&= f(\mu(p), \nu(q))(\mathrm{T}_p\mu)^{\mathrm{ad}}(\eta(\mu(p))) \otimes (\mathrm{T}_q\nu)^{-1}(\xi(\nu(q))) \\ &= ((\mu, \nu)^* f)(p, q)\mu^*\eta(p) \otimes \nu^*\xi(q). \end{aligned} \tag{A.4}$$

In case $M_1 = M_2$, $N_1 = N_2$ and $\mu = \nu$ we simply write μ^* instead of $(\mu, \mu)^*$. Note, however, that this special case is not sufficient for the purpose of this article, cf. the respective remark preceding Definition 6.5.

As can easily be checked, these pullback actions commute with the isomorphism (A.2): for any $\Upsilon \in \Gamma(\mathrm{TP}(M_2, N_2))$ we have

$$(\mu, \nu)^*(\Upsilon_\bullet) = ((\mu, \nu)^*\Upsilon)_\bullet\,. \tag{A.5}$$

Although $\mathrm{TO}(M,N)$ is a vector bundle over $M\times N$, it is important to note that an arbitrary diffeomorphism $\rho : M_1\times N_1 \to M_2\times N_2$ does not, in general, induce a natural pullback action $\rho^* : \Gamma(\mathrm{TO}(M_2,N_2)) \to \Gamma(\mathrm{TO}(M_1,N_1))$ which reduces to (A.3) for the particular case $\rho = \mu\times\nu$ as above. For a counterexample, consider the flip operator $\mathrm{fl} : M\times N \to N\times M$. An analogous statement holds for the case of two-point tensors. This is the reason why we use the notation $(\mu,\nu)^*$ for the above actions rather than $(\mu\times\nu)^*$, which would give the wrong impression of being the composition of $\mu\times\nu$ with the (non-existent) pullback operation for general diffeomorphisms alluded to above. This underlines that TP and TO have to be treated as genuine bifunctors and cannot be factorized via $(M,N)\mapsto M\times N$ composed with a single-argument functor.

Before introducing the Lie derivative of transport operators let us recollect some basic facts on the Lie derivative of smooth sections of a vector bundle E over M with respect to a smooth vector field $X\in\mathfrak{X}(M)$.

Following [25, 6.14–15], we assume that a functor F is given, assigning a vector bundle $F(M)$ over M to every manifold M of dimension n. Moreover, to every local diffeomorphism $\mu : M \to N$, the functor F assigns a vector bundle homomorphism $F(\mu) : F(M) \to F(N)$ over μ, acting as a linear isomorphism on each fiber. Given an arbitrary smooth vector field $X \in \mathfrak{X}(M)$, we assume that the local action $(\tau, v) \mapsto F(\mathrm{Fl}^X_\tau)v$ is a smooth function of (τ, v), mapping some $(-\tau_0, +\tau_0)\times F(M)|_U$ into $F(M)$, where $\tau_0 > 0$, U is an open subset of M and $\mathrm{Fl}^X_\tau : (-\tau_0,+\tau_0)\times U \to M$. Then for $\eta\in\Gamma(F(M))$, we define the pullback of η under Fl^X_τ locally by $(\mathrm{Fl}^X_\tau)^*\eta := F(\mathrm{Fl}^X_{-\tau})\circ\eta\circ\mathrm{Fl}^X_\tau$. $(\mathrm{Fl}^X_\tau)^*\eta$ being smooth on $(-\tau_0,+\tau_0)\times U$, we set $(\mathrm{L}_X\eta)(p) := \frac{\mathrm{d}}{\mathrm{d}\tau}\big|_0((\mathrm{Fl}^X_\tau)^*\eta)(p)$. Smoothness in τ (for p fixed) yields existence of $(\mathrm{L}_X\eta)(p)$ while smoothness in (τ,p) yields smoothness of the local section $\mathrm{L}_X\eta$ of $F(M)|_U$. The family of all such local sections consistently defines $\mathrm{L}_X\eta\in\Gamma(F(M))$. All the preceding applies, in particular, to $F(M) := \mathrm{T}^s_r M$ and $F(M) := \bigwedge^n \mathrm{T}^*M$.

In the following, we fix M and write E for $F(M)$.

Remark A.1. Under specific assumptions, we can say more about the action of flows resp. about Lie derivatives:

(i) If X is complete, we can take $U = M$ in the above which renders Fl^X_τ, $F(\mathrm{Fl}^X_\tau)$ and $(\mathrm{Fl}^X_\tau)^*\eta$ (smooth and) defined globally on $\mathbb{R}\times M$ resp. $\mathbb{R}\times E$ resp. $\mathbb{R}\times\Gamma(E)$. In this case, $\frac{1}{\tau}[(\mathrm{Fl}^X_\tau)^*\eta-\eta]$ tends to $\mathrm{L}_X\eta$ as $\tau\to 0$ in the linear space $\Gamma(E)$ with respect to the topology of pointwise convergence on M.

(ii) If $\operatorname{supp}\eta$ is compact there exists τ_0 such that a local version of $(\mathrm{Fl}^X_\tau)^*\eta$ can be extended from $(-\tau_0,+\tau_0)\times U$ to $(-\tau_0,+\tau_0)\times M$ by values 0.

In this case, $\frac{1}{\tau}[(\mathrm{Fl}_\tau^X)^*\eta - \eta]$ tends to $\mathrm{L}_X\eta$ as $\tau \to 0$ in the linear space $\Gamma_\mathrm{c}(E)$ with respect to the topology of pointwise convergence on M.

In the sequel, $\Gamma(E)$ and $\Gamma_\mathrm{c}(E)$ will always be equipped with the (F)- resp. the (LF)-topology. The following Proposition strengthens the statements of the preceding remark, making essential use of calculus in convenient vector spaces (cf. Appendix B).

Proposition A.2. *Let F be a functor as specified above and let $X \in \mathfrak{X}(M)$.*

(1) Assume X to be complete. Then

- *(i) $(\tau, \eta) \mapsto (\mathrm{Fl}_\tau^X)^*\eta$ is smooth from $\mathbb{R} \times \Gamma(E)$ into $\Gamma(E)$.*
- *(ii) $(\tau, \omega) \mapsto (\mathrm{Fl}_\tau^X)^*\omega$ is smooth from $\mathbb{R} \times \Gamma_\mathrm{c}(E)$ into $\Gamma_\mathrm{c}(E)$.*
- *(iii) $\mathrm{L}_X\eta = \lim\limits_{\tau\to 0} \frac{1}{\tau}[(\mathrm{Fl}_\tau^X)^*\eta - \eta]$ in $\Gamma(E)$, for every $\eta \in \Gamma(E)$.*

(2) Let X be arbitrary, $K \subset\subset M$ and $\tau_0 > 0$ such that $(\mathrm{Fl}_\tau^X)^*\omega$ is defined for all $\omega \in \Gamma_{\mathrm{c},K}(E)$ and all $|\tau| \leq \tau_0$. Then

- *(i) $(\tau, \omega) \mapsto (\mathrm{Fl}_\tau^X)^*\omega$ is smooth from $(-\tau_0, +\tau_0) \times \Gamma_{\mathrm{c},K}(E)$ into $\Gamma_\mathrm{c}(E)$.*
- *(ii) $\mathrm{L}_X\omega = \lim\limits_{\substack{\tau\to 0\\ |\tau|<\tau_0}} \frac{1}{\tau}[(\mathrm{Fl}_\tau^X)^*\omega - \omega]$ in $\Gamma_\mathrm{c}(E)$, for every $\omega \in \Gamma_{\mathrm{c},K}(E)$.*

Proof. (1) $(\tau, \eta, p) \mapsto \eta(\mathrm{Fl}_\tau^X p) = \mathrm{ev}(\eta, \mathrm{Fl}_\tau^X p)$ is smooth, due to smoothness of Fl^X and of $\mathrm{ev} : \Gamma(E) \times M \to E$ (the latter follows from the definition of the (C)-topology on spaces of smooth sections, cf. [26, 30.1]). By our assumptions on F, also $\phi : (\tau, \eta, p) \mapsto F(\mathrm{Fl}_{-\tau}^X)\eta(\mathrm{Fl}_\tau^X p) = ((\mathrm{Fl}_\tau^X)^*\eta)(p)$ is smooth; ϕ can be viewed as a section of the vector bundle $\mathrm{pr}_3^*\Gamma(E)$ over $\mathbb{R} \times \Gamma(E) \times M$. Applying Corollary B.11 we obtain that $\phi^\vee$ is a smooth map from $\mathbb{R} \times \Gamma(E)$ into $\Gamma(E)$, which is (i). (iii) now follows immediately by fixing η. In order to establish (ii), replace $\eta \in \Gamma(E)$ by $\omega \in \Gamma_\mathrm{c}(E)$ in the proof of (i), yielding smoothness of $(\tau, \omega) \mapsto (\mathrm{Fl}_\tau^X)^*\omega$ into $\Gamma(E)$. Since Fl^X maps each set of the form $[-\tau_0, +\tau_0] \times K$ (with $K \subset\subset M$) onto some compact subset L of M we see that $\phi^\vee([-\tau_0, +\tau_0] \times \Gamma_{\mathrm{c},K}(E))$ is contained in $\Gamma_{\mathrm{c},L}(E)$. A slight generalization of [18, Th. 2.2.1] establishes smoothness of $\phi^\vee$ as a map from $\mathbb{R} \times \Gamma_\mathrm{c}(E)$ into $\Gamma_\mathrm{c}(E)$. (An alternative argument completing the proof of (ii) exploits linearity in ω by passing to $\phi^{\vee\vee} : \Gamma_{\mathrm{c},K}(E) \to \mathcal{C}^\infty((-\tau_0, +\tau_0), \Gamma_\mathrm{c}(E))$ via the exponential law from [26, 27.17]).

(2) The proof of (i) is similar to the proof of (ii) of part (1): Just replace $\mathbb{R}$ by $(-\tau_0, +\tau_0)$ and $\Gamma_\mathrm{c}(E)$ by $\Gamma_{\mathrm{c},K}(E)$ in the domain of the respective maps (note that τ_0 depends on K, forcing us to restrict statements (i) and (ii) to the subspace $\Gamma_{\mathrm{c},K}(E)$ of $\Gamma_\mathrm{c}(E)$). (ii) again follows from (i). $\square$

As the respective proofs show, "local" variants of (1)(i) and (1)(iii) of the preceding result hold for arbitrary vector fields X, in the following sense: Denoting by U_τ the (open) set $\{p \in M \mid \mathrm{Fl}^X_\tau(p) \text{ and } \mathrm{Fl}^X_{-\tau}(p) \text{ are defined}\}$, the map $(\tau, \eta) \mapsto (\mathrm{Fl}^X_\tau)^*\eta$ is smooth from $(-\tau_0, +\tau_0) \times \Gamma(E)$ into $\Gamma(E|_{U_{\tau_0}})$ and $\mathrm{L}_X\eta$ exists as the respective limit in $\Gamma(E|_{U_{\tau_0}})$, both for τ_0 small enough as to make U_{τ_0} nonempty. However, there is no local variant of (1)(ii) as a map from, say, $(-\tau_0, +\tau_0) \times \Gamma_\mathrm{c}(E)$ into $\Gamma_\mathrm{c}(E|_{U_{\tau_0}})$: For $(\mathrm{Fl}^X_\tau)^*\omega$ to have compact support in U_{τ_0} for all $|\tau| < \tau_0$ we would have to assume $\operatorname{supp}\omega \subseteq \bigcap_{|\tau|<\tau_0} \mathrm{Fl}^X_\tau(U_{\tau_0})$ which excludes $(-\tau_0, +\tau_0) \times \Gamma_\mathrm{c}(E)$ as domain of the map envisaged above.

We will express statements (1)(iii) and (2)(ii) of Proposition A.2 by saying that $\mathrm{L}_X\eta$ exists in the (F)-sense resp. that $\mathrm{L}_X\omega$ exists in the (LF)-sense.

Turning now to the definition of the Lie derivative for transport operators and two-point tensors we have to extend the setting of [25], capable of handling only single argument functors F as outlined above, to bifunctors G (such as TP and TO), assigning a vector bundle $G(M, N)$ over $M \times N$ to each pair M, N of manifolds and a vector bundle isomorphism $G(\mu, \nu) : G(M_1, N_1) \to G(M_2, N_2)$ over $\mu \times \nu$ to every pair of local diffeomorphisms $\mu : M_1 \to M_2$, $\nu : N_1 \to N_2$. *Mutatis mutandis*, all statements of Remark A.1 and Proposition A.2 remain valid for bifunctors of that type. Thus, let $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ be complete vector fields with flows Fl^X and Fl^Y, respectively. We then define the Lie derivative by differentiating at 0 the pullback (under the flow of (X, Y)) of any given transport operator $A \in \Gamma(\mathrm{TO}(M, N))$:

$$\mathrm{L}_{X,Y} A(p, q) := \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} (\mathrm{Fl}^X_\tau, \mathrm{Fl}^Y_\tau)^* A(p, q)\,. \tag{A.6}$$

Analogously, for $\Upsilon = f\eta \otimes \xi$ a generic element of $\Gamma(\mathrm{TP}(M, N))$ we set

$$\begin{aligned} \mathrm{L}_{X,Y}\Upsilon(p, q) &= \tfrac{\mathrm{d}}{\mathrm{d}\tau}\big|_0 (\mathrm{Fl}^X_\tau, \mathrm{Fl}^Y_\tau)^*\Upsilon(p, q) \\ &= (\mathrm{L}_{X,0} f + \mathrm{L}_{0,Y} f)(p, q)\, \eta(p) \otimes \xi(q) \\ &\quad + f(p, q)(\mathrm{L}_X\eta(p) \otimes \xi(q) + \eta(p) \otimes \mathrm{L}_Y\xi(q)). \end{aligned} \tag{A.7}$$

From (A.5), we obtain for any $\Upsilon \in \Gamma(\mathrm{TP}(M, N))$:

$$\mathrm{L}_{X,Y}(\Upsilon_\bullet) = (\mathrm{L}_{X,Y}\Upsilon)_\bullet\,. \tag{A.8}$$

Resuming the discussion of the pullback action of *pairs* of diffeomorphism started after (A.5) we see that also in order to implement a geometric approach to Lie derivatives via pullback action of flows, definitions (A.6) and (A.7) had to be based on pairs of flows $(\mathrm{Fl}^X_\tau, \mathrm{Fl}^Y_\tau)$ on M resp. N rather than

on the flow of some single vector field Z on $M \times N$. This is emphasized by our notation $\mathrm{L}_{X,Y}$ rather than $\mathrm{L}_{X\times Y}$, reflecting the fact that it is precisely the vector fields of the form $Z = (X, Y)$ from the subspace $\mathfrak{X}(M) \oplus \mathfrak{X}(N)$ of $\mathfrak{X}(M \times N)$ that induce a pullback action and a Lie derivative on (sections of) the bundle functors TO and TP. In this sense, our concept of Lie derivative is, in fact, a proper extension resp. refinement of the usual setting as presented, e.g., in [25] where only single-argument (vector bundle valued) functors are considered.

Appendix B Auxiliary results from calculus in convenient vector spaces

The notion of a smooth curve $c : \mathbb{R} \to E$ where E is some locally convex space is unambiguous. The space $\mathcal{C}^\infty(\mathbb{R}, E)$ of smooth curves in E will always carry the topology of uniform convergence on compact intervals in all derivatives separately. For locally convex spaces E, F, a map $f : E \to F$ is defined to be smooth if $c \mapsto f \circ c$ takes smooth curves in E to smooth curves in F. This notion of smoothness depends only on the respective families of bounded sets, i.e., if the topologies of E and F are changed in such a way that in each space the family of bounded subsets is preserved then the space $\mathcal{C}^\infty(E, F)$ of smooth mappings from E to F remains the same ([26, 1.8]). As a rule, we endow $\mathcal{C}^\infty(E, F)$ with the "(C)-topology" ("C" standing for "curve" resp. "$\mathcal{C}^\infty$" resp. "convenient"), defined as the initial (locally convex) topology with respect to the family of all mappings $c^* : \mathcal{C}^\infty(E, F) \to \mathcal{C}^\infty(\mathbb{R}, F)$, for $c \in \mathcal{C}^\infty(\mathbb{R}, E)$ ([26, 3.11]). The evaluation map $\mathrm{ev} : \mathcal{C}^\infty(E, F) \times E \to F$ sending (f, x) to $f(x)$ is smooth by [26, 3.13 (1)]. Consequently, $\mathrm{ev}_x : \mathcal{C}^\infty(E, F) \ni f \mapsto f(x) \in F$ is (linear and) smooth (equivalently, bounded, due to [26, 2.11]) for every fixed $x \in E$.

In the above, E can be replaced by some open (even c^∞-open, cf. [26, 2.12]) subset U of E resp. by some smooth and smoothly Hausdorff (cf. [26, p. 265]) manifold M modelled over convenient vector spaces ([26, 27.17]; as to the evaluation map, see the proof of Lemma B.9).

There are many equivalent ways how to define a convenient vector space, cf. [26, 2.14, Th.]. We will use condition (6) of this theorem saying that a locally convex space E is convenient if for each bounded absolutely convex closed subset B, the normed space (E_B, p_B) is complete. Every sequentially complete locally convex vector space is convenient ([26, 2.2]); all the spaces considered in this article are sequentially complete.

Note that, in general, smooth maps are not necessarily continuous. If,

however, E is metrizable then any $f \in \mathcal{C}^\infty(E, F)$ is continuous, due to [26, 4.11, 2.12, and p. 8].

If E and F have the property that smooth maps $f : E \to F$ map compact sets to bounded sets (in particular, if E is metrizable or an (LF)-space), there is a second natural locally convex topology on $\mathcal{C}^\infty(E, F)$ which will be called (D)-topology: This is the topology of convergence of differentials (hence "D") of all orders l (separately), uniformly on sets of the form $K \times B^l$ where K is a compact and B a bounded subset of E. By the chain rule ([26, 3.18]), the (D)-topology is finer than the (C)-topology. In fact, even on $\mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$, an inspection of the form of the respective typical neighborhoods of 0 reveals (D) to be strictly finer than (C). However, Theorem B.1 below shows that (D) and (C) have the same bounded sets if E is an (F)-space or an (LF)-space. Therefore, the notion of smoothness on $\mathcal{C}^\infty(E, F)$ with respect to both topologies is the same in that case. On $\mathcal{C}^\infty(\mathbb{R}, E)$, the (D)-topology coincides with the usual (F)-topology.

For the proof of the theorem below, note the following: Calling a sequence $x_n \to x$ in a topological vector space fast converging if for each $l \in \mathbb{N}$, the sequence $n^l(x_n - x)$ tends to 0, then every convergent sequence in a metrizable topological vector space or an (LF)-space possesses a fast converging subsequence. To see this, let $(V_k)_k$ be a decreasing neighborhood base of 0 consisting of circled sets and choose $n_k \in \mathbb{N}$ monotonically increasing such that $(x_{n_k} - x) \in k^{-k} V_k$.

Theorem B.1. *Let E be an (F)-space or an (LF)-space and F be an arbitrary locally convex space. Then every (C)-bounded subset of $\mathcal{C}^\infty(E, F)$ is (D)-bounded, hence (C) and (D) have the same bounded sets.*

Proof. Let B be bounded with respect to the (C)-topology, i.e.,

$$\forall c \in \mathcal{C}^\infty(\mathbb{R}, E)\ \forall K_0 \subset\subset \mathbb{R}\ \forall l \in \mathbb{N}_0 : \\ \{(f \circ c)^{(l)}(\tau) \mid \tau \in K_0,\ f \in B\} \text{ is bounded in } F.$$

Assume, by way of contradiction, B not to be bounded with respect to (D), i.e.

$$\exists K \subset\subset E\ \exists D\,(\text{bounded} \subseteq E)\ \exists l \in \mathbb{N}_0 : \\ \{\mathrm{d}^l f(p)(w, \ldots, w) \mid f \in B,\ p \in K,\ w \in D\} \text{ is unbounded in } F.$$

Here, we have already used polarization as, e.g., in [26, 7.13 (1)], to obtain equal vector arguments $(w, \ldots, w)$. Fix K, D, l as above. By the preceding, we can choose sequences $f_k \in B$, $p_k \in K$, $w_k \in D$ such that

$$\frac{1}{k^{2lk}}\,\mathrm{d}^l f_k(p_k)(w_k, \ldots, w_k) \not\to 0 \quad (k \to \infty). \tag{B.1}$$

By passing to suitable subsequences we can assume that there is $p \in K$ such that $p_k \to p$ fast as $k \to \infty$. (Note that if E is an (LF)-space, we are working within one fixed Fréchet subspace of E, due to K and D being bounded.) Setting $v_k := k^{-k} w_k$ we obtain $v_k \to 0$ fast, due to D being bounded. Now, by [26, 2.10], there are a smoothly parametrized polygon $c : \mathbb{R} \to E$ and numbers $\tau_k \to 0$ in $\mathbb{R}$ such that $c(\tau_k + \tau) = p_k + \tau v_k$ for $\tau \in (-\delta_k, +\delta_k)$, for suitable $\delta_k \in (0,1)$.

Recalling that B is (C)-bounded, choose a compact interval $K_0 \subset\subset \mathbb{R}$ containing all intervals $(\tau_k - \delta_k, \tau_k + \delta_k)$ and take l and c as above. Then we conclude that $\{(f \circ c)^{(l)}(\tau) \mid \tau \in K_0,\ f \in B\}$ is bounded in F. Defining $y_k := (f_k \circ c)^{(l)}(\tau_k) = \mathrm{d}^l f_k(p_k)(v_k, \ldots, v_k)$, we therefore have $k^{-lk} y_k \to 0$. Consequently,

$$\frac{1}{k^{2lk}}\, \mathrm{d}^l f_k(p_k)(w_k, \ldots, w_k) \quad = \quad \frac{1}{k^{lk}}\, y_k \to 0,$$

contradicting (B.1). $\square$

Note that the preceding theorem remains valid for $\mathcal{C}^\infty(U,F)$ where U is an open subset of the (F)- resp. (LF)-space E: The relevant part of the polygon c constructed in the proof can be assumed to be contained in some given absolutely convex neighborhood of p.

In order to carry over Theorem B.1 to spaces $\Gamma(M,E)$ of smooth sections of vector bundles $E \xrightarrow{\pi} M$, we need a suitable notion of (C)-topology on section spaces $\Gamma(M,E)$, E having some convenient vector space Z as typical fiber. It certainly would be tempting to proceed as follows ([26, p. 294]): For any local trivialization (U,Ψ) on $E \xrightarrow{\pi} M$ $(\Psi : \pi^{-1}U \to U \times Z)$ and $u \in \Gamma(M,E)$, let $\tilde{\Psi}(u) \in \mathcal{C}^\infty(U,Z)$ be given by

$$\tilde{\Psi}(u) := \mathrm{pr}_2 \circ \Psi \circ (u|_U).$$

Now we could define the (C)-topology on $\Gamma(M,E)$ as the initial topology with respect to the family of all maps $\{\tilde{\Psi}_\alpha \mid \alpha \in A\} : \Gamma(M,E) \to \mathcal{C}^\infty(U_\alpha, Z)$ where $\{(U_\alpha, \Psi_\alpha) \mid \alpha \in A\}$ is any vector bundle atlas[2] for $E \xrightarrow{\pi} M$. Here, $\mathcal{C}^\infty(U_\alpha, Z)$, in turn, carries the (C)-topology in the sense of [26, 27.17]. This construction, however, *does* depend on the atlas used: Already for the trivial bundle $\mathbb{R} \times Z$ with atlas $\{\mathrm{id}_{\mathbb{R}\times Z}\}$, the (C)-topology on $\Gamma(\mathbb{R}, \mathbb{R} \times Z)$ would become strictly finer by adjoining the local trivialization $\mathrm{id}_{\mathbb{R}} \times \phi$ where ϕ is a discontinuous bornological isomorphism of Z. It is not hard to see that the (C)-topology, as defined above, does not depend on the atlas if Z is barreled

[2]The term vector bundle atlas denotes a compatible family $\{(U_\alpha, \Psi_\alpha) \mid \alpha \in A\}$ of local trivializations on $E \xrightarrow{\pi} M$ with $\bigcup_\alpha U_\alpha = M$.

and bornological. Thus, we could either accept this additional condition or define the (C)-topology via the maximal vector bundle atlas. For the present purposes, however, only the family of (C)-bounded subsets is relevant which, fortunately, is independent of the atlas: It suffices to note that for $w \in \mathcal{C}^\infty(U_{\alpha\beta}, Z)$, the "change of vector bundle chart" $w \mapsto \mathrm{ev} \circ (\psi_{\alpha\beta} \times w) \circ \Delta$ (where $\Delta : U_\alpha \cap U_\beta =: U_{\alpha\beta} \to U_{\alpha\beta} \times U_{\alpha\beta}$ denotes the diagonal map and $\psi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(Z)$ the transition functions) is linear and smooth by [26, 3.13 (1)(6)(7)], hence bounded (this substantiates and extends the respective remarks preceding the proposition in [26, p. 294]).

Corollary B.2. *Let $E \xrightarrow{\pi} M$ be a vector bundle over M (with $\dim M$ and $\dim E$ finite). Then a subset of $\Gamma(M, E)$ is (C)-bounded if and only if it is (F)-bounded.*

Proof. We may assume that every local trivialization (U_α, Ψ_α) as above is defined over some chart (U_α, ψ_α) of M. Let B denote a subset of $\Gamma(M, E)$. Then the following statements are equivalent ($\alpha \in A$ as above):

(i) B is (C)-bounded.

(ii) $\tilde{\Psi}_\alpha(B)$ is (C)-bounded in $\mathcal{C}^\infty(U_\alpha, Z)$, for every α.

(iii) $(\psi_\alpha^{-1})^* \tilde{\Psi}_\alpha(B)$ is (C)-bounded in $\mathcal{C}^\infty(\psi_\alpha(U_\alpha), Z)$, for every α.

(iv) $(\psi_\alpha^{-1})^* \tilde{\Psi}_\alpha(B)$ is (D)-bounded in $\mathcal{C}^\infty(\psi_\alpha(U_\alpha), Z)$, for every α.

(v) All seminorms generating the (F)-topology on $\Gamma(M, E)$ (cf. (2.1)) are bounded on B.

(vi) B is (F)-bounded.

The equivalences of (i)–(iv) are immediate from the definition of the (C)-topologies on $\Gamma(M, E)$, resp. on $\mathcal{C}^\infty(U_\alpha, Z)$, resp. from Theorem B.1, in turn (note that $\psi_\alpha(U_\alpha)$ is an open subset of the (F)-space $\mathbb{R}^{\dim M}$). Taking into account $((\psi_\alpha^{-1})^* \circ \tilde{\Psi}_\alpha)(u) = \mathrm{pr}_2 \circ \Psi_\alpha \circ (u|_{U_\alpha}) \circ \psi_\alpha^{-1} = (\psi_\alpha^j \circ (u|_{U_\alpha}) \circ \psi_\alpha^{-1})_{j=1}^{\dim E}$, (2.1) yields (iv)$\Leftrightarrow$(v). (v)$\Leftrightarrow$(vi), finally, holds by definition. $\square$

Remark B.3. Corollary B.2 shows, in particular, that smoothness of members of (2) resp. (3) in Lemma 7.2 is not affected by replacing the (C)- by the (F)-topology on $\mathcal{T}^r_s(M)$ resp. on $\mathcal{C}^\infty(M)$. Due to Theorem B.1, a similar statement is true for the (C)- and the (D)-topologies on $\mathcal{C}^\infty(\mathcal{T}^s_r(M), \mathcal{C}^\infty(M))$ resp. on $\mathcal{C}^\infty(\mathcal{T}^s_r(M), \mathbb{R})$ in (3) resp. (5) of Lemma 7.2, provided that M is separable, hence $\mathcal{T}^s_r(M)$ is an (F)-space.

In the proof of the following theorem, two applications of the uniform boundedness principle as formulated in condition (2) of [26, 5.22] will occur. One of the assumptions of this principle is that for the set $B(\subseteq E)$ to be demonstrated as being bounded, the normed subspace $E_B = \bigcup_n nB$ of E (cf. [36, p. 63]) has to be a Banach space with respect to the Minkowski functional p_B of B. Lemma B.4 below makes sure that this condition is available when needed below.

Lemma B.4. *Let E, F be vector spaces, F_λ (with λ from some index set) locally convex vector spaces, $f : E \to F$ linear and $g_\lambda : F \to F_\lambda$ a family of linear maps which is assumed to be point separating on F. If B is an absolutely convex subset of E such that E_B is a Banach space, and for every λ the set $(g_\lambda \circ f)(B)$ is bounded in F_λ then also $F_{f(B)}$ is a Banach space.*

Proof. By standard methods, it follows that the Minkowski functional $p_{f(B)}$ of $f(B)$ is the quotient semi-norm of p_B on the space $F_{f(B)} \cong E_B/\ker(f|_{E_B})$. Since $(g_\lambda \circ f)(B)$ is bounded, $p_{f(B)}$ is even a norm (observe that for $p_{f(B)}(x) = 0$, $g_\lambda(x)$ is contained in the intersection of all neighborhoods of zero in F_λ). Thus $(F_{f(B)}, p_{f(B)})$, being a quotient of the Banach space (E_B, p_B), is a Banach space in its own right. □

One more technical remark is in order: Recall that a convenient vector space is a locally convex vector space in which for each bounded absolutely convex closed subset B, the normed space (E_B, p_B) is complete ([26, Th. 2.14]). Now, if we are given a linear bijection between two convenient vector spaces then, in order to show that the respective families of bounded sets are corresponding to each other, i.e., that the given map is a bornological isomorphism, it is sufficient to show that for every bounded absolutely convex closed subset B' of one of the spaces such that $E_{B'}$ is a Banach space, B' is bounded also as a subset of the other space: Indeed, if B is an arbitrary bounded subset of the first space then for its absolutely convex closed hull $B' = \overline{\Gamma}(B)$, $E_{B'}$ is Banach. By assumption, B', hence *a fortiori* B, is bounded when viewed as a subset of the second space. Recall further that by subscripts "F" resp. "C" we declare the respective space as being equipped with the (F)- resp. the (C)-topology.

Theorem B.5. *For every finite-dimensional manifold M, the linear isomorphism $\phi : t \mapsto (\tilde{t} \mapsto t \cdot \tilde{t})$ mapping $\mathcal{T}^r_s(M)$ onto $\mathrm{L}^b_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M)_F, \mathcal{C}^\infty(M)_C)$ is a bornological isomorphism with respect to the (C)-topologies.*

Proof. That ϕ indeed is a linear isomorphism was shown in the proof of Lemma 7.2. Let B be a subset of $\mathcal{T}^r_s(M)$. The proof will be achieved by showing (partially under a certain additional assumption on B, cf. below)

the mutual equivalence of the following statements, where always $p \in M$ and $\tilde{t} \in \mathcal{T}^s_r(M)$:

(i) B is bounded with respect to the (C)-topology of $\mathcal{T}^r_s(M)$.

(ii) $\{t(p) \mid t \in B\}$ is bounded in $(\mathrm{T}^r_s)_p M$, for every p.

(iii) $\{t(p) \cdot \tilde{t}(p) \mid t \in B\}$ is bounded in $\mathbb{R}$, for every $p, \tilde{t}$.

(iv) $\{\phi(t)(\tilde{t})(p) \mid t \in B\}$ is bounded in $\mathbb{R}$, for every $p, \tilde{t}$.

(v) $\{\phi(t)(\tilde{t}) \mid t \in B\}$ is (C)-bounded in $\mathcal{C}^\infty(M)$, for every $\tilde{t}$.

(vi) $\phi(B)$ is bounded w.r.t the (C)-topology of $\mathrm{L}^b_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M)_F, \mathcal{C}^\infty(M)_C)$.

Identifying (iii)$\Leftrightarrow$(iv) as a mere reformulation and discerning the chains (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii) and (iv)$\Leftarrow$(v)$\Leftarrow$(vi) as being obvious (evaluation at a particular argument always being smooth and linear, hence bounded, essentially due to [26, 3.13 (1) and 2.11]), we are left with three non-trivial implications. Turning to (v)$\Rightarrow$(vi), first of all note that the (C)-bounded subsets of $\mathrm{L}^b_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{C}^\infty(M))$ (omitting from now on the subscripts "F" resp. "C") can equivalently be viewed as those determined by the structures of $\mathcal{C}^\infty(\mathcal{T}^s_r(M), \mathcal{C}^\infty(M))$ resp. of $\mathrm{L}^b(\mathcal{T}^s_r(M), \mathcal{C}^\infty(M))$, due to [26, 5.3. Lemma] (where the superscript "b" is omitted generally). Now an appeal to the uniform boundedness principle for spaces of (multi)linear mappings ([26, 5.18. Th.]) yields (v)$\Rightarrow$(vi).

Whereas the implications established so far are valid for any subset B of $\mathcal{T}^r_s(M)$, we will have to confine ourselves for (ii)$\Rightarrow$(i) and (iv)$\Rightarrow$(v) to subsets B which are absolutely convex and for which $\mathcal{T}^r_s(M)_B$ is a Banach space. This being a purely algebraic matter, it is equivalent to saying that $\phi(B)$ is absolutely convex and $\mathrm{L}^b_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{C}^\infty(M))_{\phi(B)}$ is a Banach space. Under this additional assumption (in its first form), we obtain (ii)$\Rightarrow$(i) from a straightforward application of the uniform boundedness principle for section spaces ([26, 30.1 Prop.]). For a similar argument in favor of (iv)$\Rightarrow$(v) to be legitimate, however, we need to know that also $\mathcal{C}^\infty(M)_{\phi(B)(\tilde{t})}$ is a Banach space, for every $\tilde{t} \in \mathcal{T}^s_r(M)$. Yet this follows— assuming (iv) to be true— from $\mathrm{L}^b_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{C}^\infty(M))_{\phi(B)}$ being a Banach space by applying Lemma B.4 with E, F, F_λ, f, g_λ replaced by $\mathrm{L}^b_{\mathcal{C}^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{C}^\infty(M))$, $\mathcal{C}^\infty(M)$, $\mathbb{R}$, $\mathrm{ev}_{\tilde{t}}$, ev_p, respectively. Now we are in a position to appeal to [26, 30.1 Prop.] once more, completing the proof of equivalence of (i)–(vi) for subsets B as specified. Keeping in mind the technical remark made before the theorem, we conclude that, in fact, ϕ is a bornological isomorphism. $\square$

Remark B.6. By Corollary B.2, the bounded sets with respect to the topologies (C) and (F) on $\mathcal{T}^r_s(M)$ are identical. Hence Theorem B.5 could have been as well formulated for $\mathcal{T}^r_s(M)_C$ replacing $\mathcal{T}^r_s(M)_F$.

Lemma B.7. *For a vector bundle $E \xrightarrow{\pi} B$, a manifold B' and a smooth map $f : B' \to B$ (B, B' and E finite-dimensional), the pullback operator $f^* : \Gamma(B,E) \to \Gamma(B', f^*E)$ defined by $f^*(u)(p) := (p, u(f(p)))$ is continuous with respect to the (F)-topologies.*

Proof. This is clear from combining the seminorms (2.1) with the typical form of charts on pullback bundles (cf. Appendix A): Given $l \in \mathbb{N}_0$, a chart (V, φ) in B', a vector bundle chart (U, Ψ) over (U, ψ) in E such that $V \cap f^{-1}(U) \neq \emptyset$ and $L \subset\subset \varphi(V \cap f^{-1}(U))$, the values $p_{s,\varphi\times_B\Psi,L}(f^*(u))$ are dominated by $p_{s,\Psi,L_1}(u)$ where $L_1 = (\psi \circ f \circ \varphi^{-1})(L)$. $\square$

Also the following Lemma is immediate from the definition of the seminorms on spaces of sections:

Lemma B.8. *For finite-dimensional manifolds M, N, the operator*

$$\mathrm{ev}^s_r : \Gamma(\mathrm{TO}(M,N)) \times \Gamma(\mathrm{pr}_1^*\mathrm{T}^s_r M) \to \Gamma(\mathrm{pr}_2^*\mathrm{T}^s_r N)$$

given by

$$(A, \xi) \mapsto \Big((p,q) \mapsto \big((p,q)\,,\, A^s_r(p,q)\,\xi(p,q)\big)\Big)$$

is continuous with respect to the (F)-topologies.

For our final lemma and its corollaries, we will have to include infinite-dimensional manifolds modelled over convenient vector spaces into our considerations, as well as bundles over such manifolds and respective spaces of sections. Again we follow [26], this time Sections 27–30. Note that the discussion concerning the (C)-topology following Theorem B.1 is equally valid for M and E infinite-dimensional. The main examples of infinite-dimensional manifolds occurring in the present context are $\hat{\mathcal{A}}_0(M) \times M \times \hat{\mathcal{B}}(M)$ and $\hat{\mathcal{A}}_0(M) \times \hat{\mathcal{B}}(M)$, where $\hat{\mathcal{A}}_0(M)$ is a closed affine hyperplane in the (LF)-space $\Omega^n_c(M)$ and $\hat{\mathcal{B}}(M)$ is an (LF)-space itself. Note that (F)-spaces, (LF)-spaces and closed hyperplanes thereof are convenient by [26, 2.2 and 2.14]. The construction of pullback bundles in [26, 29.6], proceeds in complete analogy to the finite-dimensional case: In particular, for $\mathrm{pr}_2 : \hat{\mathcal{A}}_0(M) \times M \times \hat{\mathcal{B}}(M) \to M$, the pullback bundle $\mathrm{pr}_2^*\mathrm{T}^s_r M$ can be realized as a manifold by $(\hat{\mathcal{A}}_0(M) \times M \times \hat{\mathcal{B}}(M)) \times_M \mathrm{T}^s_r M$, or—more simply—by $\hat{\mathcal{A}}_0(M) \times \hat{\mathcal{B}}(M) \times \mathrm{T}^s_r M$, with base point map $(\omega, A, t) \mapsto (\omega, \pi(t), A)$ (the version which we will exclusively use in what follows).

As a last prerequisite to the following lemma, we observe that the evaluation map $\mathrm{ev} : \mathcal{C}^\infty(M,E) \times M \to E$ is smooth for any convenient vector space E and for every manifold M. This result (which does not appear explicitly in [26]) can be obtained by an argument completely analogous to that of the vector space case (where an open subset U of a locally convex space E takes the place of M): Due to the proof of [26, 27.17], $\mathcal{C}^\infty(M,F)$ can be viewed as a closed linear subspace of a product of spaces $\mathcal{C}^\infty(\mathbb{R},F)$. This statement replacing [26, 3.11 Lemma] in the proof of [26, Th. 3.12], the latter as well as Cor. 3.13 (1) (saying that $\mathrm{ev} : \mathcal{C}^\infty(U,F) \times U \to F$ is smooth) together with their proofs carry over to the manifold case. As a by-product, we obtain a fact which will be tacitly used in the proof of the lemma below: By continuity of the evaluation map and by the manifold analogue of [26, 3.12] just mentioned, it follows that the obvious linear isomorphism $\mathcal{C}^\infty(M,\prod_\alpha E_\alpha) \cong \prod_\alpha \mathcal{C}^\infty(M,E_\alpha)$ (for a manifold M and convenient vector spaces E_α) is even a bornological isomorphism.

Lemma B.9. [A. Kriegl, personal communication] *Let M, N be manifolds and $E \xrightarrow{\pi} N$ a smooth vector bundle over N (M, N, E possibly infinite-dimensional). Then we have a bornological isomorphism*

$$\mathcal{C}^\infty(M,\Gamma(N,E)) \cong \Gamma(M\times N, \mathrm{pr}_2^* E)$$

with respect to the (C)-topologies. Elements f, g corresponding to each other by this isomorphism are related by $(p, f(p)(q)) = g(p,q)$; we write $g = f^\wedge$, $f = g^\vee$. Moreover, the evaluation mapping

$$\mathrm{ev} : \mathcal{C}^\infty(M,\Gamma(N,E)) \times M \to \Gamma(N,E)$$

is smooth with respect to the (C)-topologies resp. the structure given on M.

Proof. For the proof, we will represent each of the two spaces as a closed (with respect to the c^∞-topology, cf. [26, 2.12]) subspace of a respective product space; these product spaces are then seen to be isomorphic by means of the exponential law for spaces of type $\mathcal{C}^\infty(P,F)$ (P a manifold and F convenient; [26, 27.17]). In all three steps, the families of bounded sets are preserved. Finally, it is shown that under the isomorphism between the product spaces in fact the subspaces $\mathcal{C}^\infty(M,\Gamma(N,E))$ and $\Gamma(M\times N,\mathrm{pr}_2^*E)$ correspond to each other.

Choose a vector bundle atlas for $E \xrightarrow{\pi} N$ consisting of local trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times Z$ where Z denotes the typical fiber of E. By [26, 30.1] we obtain a linear embedding

$$\Gamma(N,E) \hookrightarrow \prod_\alpha \mathcal{C}^\infty(U_\alpha, Z)$$

$$((\mathrm{pr}_2 \circ \psi_\alpha)_*)_\alpha : u \mapsto (\mathrm{pr}_2 \circ \psi_\alpha \circ u|_{U_\alpha})_\alpha$$

having c^∞-closed image, the latter due to the fact that that it can be characterized as the subspace consisting of all $(f_\alpha)_\alpha$ for which the maps $q \mapsto \psi_\alpha^{-1}(q, f_\alpha(q))$ form a coherent family of local sections of E. For c^∞-closedness, express coherence by the conditions $\phi_{\alpha\beta}(f_\alpha) = \phi_{\beta\alpha}(f_\beta)$ for all α, β where

$$\phi_{\alpha\beta} : \mathcal{C}^\infty(U_\alpha, Z) \ni f_\alpha \mapsto \psi_\alpha^{-1} \circ (\mathrm{id}_{U_\alpha}, f_\alpha)|_{U_{\alpha\beta}} \in \Gamma(U_{\alpha\beta}, E|U_{\alpha\beta})$$

and similarly for $\phi_{\beta\alpha}$. Now, it certainly would suffice to show smoothness (hence c^∞-continuity) of both $\phi_{\alpha\beta}$ and $\phi_{\beta\alpha}$. Making things work for $\phi_{\alpha\beta}$ requires $\Gamma(U_{\alpha\beta}, E|U_{\alpha\beta})$ to be equipped with the topology induced by the atlas $\mathcal{A}_\alpha := \{(U_{\alpha\beta}, \psi_\alpha|_{E|U_{\alpha\beta}})\}$ on $E|U_{\alpha\beta}$ whereas for $\phi_{\beta\alpha}$, the atlas $\mathcal{A}_\beta := \{(U_{\alpha\beta}, \psi_\beta|_{E|U_{\alpha\beta}})\}$ would be appropriate.

However, due to the remarks preceding Corollary B.1, we can safely use $\mathcal{A}_\alpha \cup \mathcal{A}_\beta$ on $E|U_{\alpha\beta}$ (doing justice to both $\phi_{\alpha\beta}$ and $\phi_{\beta\alpha}$ simultaneously this way) without affecting boundedness in $\Gamma(U_{\alpha\beta}, E|U_{\alpha\beta})$ resp. smoothness of $\phi_{\alpha\beta}$ and $\phi_{\beta\alpha}$.

From the above, we get an embedding

$$\mathcal{C}^\infty(M, \Gamma(N, E)) \hookrightarrow \mathcal{C}^\infty\Big(M, \prod_\alpha \mathcal{C}^\infty(U_\alpha, Z)\Big) \cong \prod_\alpha \mathcal{C}^\infty(M, \mathcal{C}^\infty(U_\alpha, Z))$$
$$((\mathrm{pr}_2 \circ \psi_\alpha)_{**})_\alpha : f \mapsto (p \mapsto \mathrm{pr}_2 \circ \psi_\alpha \circ f(p)|_{U_\alpha})_\alpha,$$

again with c^∞-closed image, consisting of those $(g_\alpha)_\alpha$ for which $\psi_\alpha^{-1}(q, g_\alpha(p)(q))$ forms a coherent family of local sections of E for every fixed $p \in M$.

On the other hand, noting the family of all $(M \times U_\alpha, \mathrm{id}_M \times \psi_\alpha)$ to form a vector bundle atlas for $\mathrm{pr}_2^* E \to M \times N$, we obtain a linear embedding

$$\Gamma(M \times N, \mathrm{pr}_2^* E) \hookrightarrow \prod_\alpha \mathcal{C}^\infty(M \times U_\alpha, Z)$$
$$\tilde{u} \mapsto (\mathrm{pr}_2 \circ (\mathrm{id}_M \times \psi_\alpha) \circ \tilde{u}|_{M \times U_\alpha})_\alpha$$

mapping $\Gamma(M \times N, \mathrm{pr}_2^* E)$ onto the c^∞-closed subspace consisting of all $(\tilde{f}_\alpha)_\alpha$ for which $(\mathrm{id}_M \times \psi_\alpha)^{-1}(p, q, \tilde{f}_\alpha(p, q)) = (p, \psi_\alpha^{-1}(q, \tilde{f}_\alpha(p, q)))$ forms a coherent family of local sections of $\mathrm{pr}_2^* E$. Via the bornological isomorphisms $\mathcal{C}^\infty(M \times U_\alpha, Z) \cong \mathcal{C}^\infty(M, \mathcal{C}^\infty(U_\alpha, Z))$ given by the exponential law [26, 27.17], we obtain an isomorphism of the last two product spaces occurring above. This, in turn, induces bornological isomorphisms of the two aforementioned subspaces resp. of $\mathcal{C}^\infty(M, \Gamma(N, E))$ and $\Gamma(M \times N, \mathrm{pr}_2^* E)$. Tracing all the assignments involved in the construction shows the explicit form of this last isomorphism to be the one given in the lemma.

Finally, replacing E by $\Gamma(N, E)$ in the remarks on smoothness of the evaluation map preceding the lemma shows that also $\mathrm{ev} : \mathcal{C}^\infty(M, \Gamma(N, E)) \times$

$M \to \Gamma(N, E)$ is smooth with respect to the (C)-topologies resp. the structure given on M. $\square$

Note that, using the notations of the preceding lemma, sections u of $\mathrm{pr}_2^* E$ are precisely given by smooth maps $\bar{u} : M \times N \to E$ with $\pi \circ \bar{u} = \mathrm{pr}_2$, i.e., $\bar{u}(p, q)$ having q as base point, for all p, q. The section u itself takes the form $u(p, q) = (p, \bar{u}(p, q))$.

For the corollary to follow, recall that $\mathrm{pr}_2{}' : \mathrm{pr}_2^* E \to E$ denotes the canonical projection as defined in Appendix A.

Corollary B.10. *For manifolds M, N and a vector bundle $E \xrightarrow{\pi} N$ (M, N, E as in Lemma B.9), the operator $\overline{\mathrm{ev}} : \Gamma(M \times N, \mathrm{pr}_2^* E) \times M \to \Gamma(N, E)$ defined by $\overline{\mathrm{ev}}(u, p)(q) := \mathrm{pr}_2{}'(u(p, q))$ is smooth when both section spaces are equipped with their (C)-topologies.*

Proof. According to Lemma B.9, $\overline{\mathrm{ev}}$ and ev correspond to each other via the bornological isomorphism $\mathcal{C}^\infty(M, \Gamma(N, E)) \cong \Gamma(M \times N, \mathrm{pr}_2^* E)$. By [26, 2.11], this isomorphism and its inverse are smooth. Hence the smoothness of $\overline{\mathrm{ev}}$ follows from that of ev. $\square$

Corollary B.11. *Let M and N be manifolds, and $E \xrightarrow{\pi} N$ a smooth vector bundle over N (N and E finite-dimensional). Then for every $u \in \Gamma(M \times N, \mathrm{pr}_2^* E)$ the associated map $u^\vee : M \to \Gamma(N, E)$ is smooth with respect to the (F)-topology on $\Gamma(N, E)$.*

Proof. By Lemma B.9, $u^\vee \in \mathcal{C}^\infty(M, \Gamma(N, E))$ where $\Gamma(N, E)$ carries the (C)-topology. By Corollary B.2, the (C)- resp. the (F)-bounded subsets on $\Gamma(N, E)$ are the same. Therefore, $u^\vee$ is also smooth into $\Gamma(N, E)_F$. $\square$

If also M has finite dimension Corollary B.11 can be proved in the well-known "classical" manner, using charts. However, we need the result in the general case where M is of infinite dimension.

Acknowledgments: We would like to thank James D.E. Grant and Andreas Kriegl for helpful discussions. Work on this paper was supported by projects P16742, Y237, and P20525 of the Austrian Science Fund.

References

[1] Balasin, H. Geodesics for impulsive gravitational waves and the multiplication of distributions. *Class. Quant. Grav.*, **14**:455–462, 1997.

[2] Bourbaki, N. *Algebra I, Chapters 1–3.* Elements of Mathematics. Hermann, Addison-Wesley, Paris, Massachusetts, 1974.

[3] Clarke, C. J. S., Vickers, J. A., Wilson, J. P. Generalised functions and distributional curvature of cosmic strings. *Class. Quant. Grav.*, **13**:2485–2498, 1996.

[4] Colombeau, J. F. *New Generalized Functions and Multiplication of Distributions.* North Holland, Amsterdam, 1984.

[5] Colombeau, J. F. *Elementary Introduction to New Generalized Functions.* North Holland, Amsterdam, 1985.

[6] Colombeau, J. F. Multiplication of distributions. *Bull. Amer. Math. Soc. (N.S.)*, **23**:251–268, 1990.

[7] Colombeau, J. F. *Multiplication of Distributions. A tool in mathematics, numerical engineering and theoretical physics*, *Lecture Notes in Mathematics* **1532**. Springer, 1992.

[8] Colombeau, J. F., Meril, A. Generalized functions and multiplication of distributions on $\mathcal{C}^\infty$ manifolds. *J. Math. Anal. Appl.*, **186**:357–364, 1994.

[9] De Roever, J. W., Damsma, M. Colombeau algebras on a $\mathcal{C}^\infty$-manifold. *Indag. Mathem., N.S.*, **2**(3), 1991.

[10] Delcroix, A. Kernel theorems in spaces of tempered generalized functions. *Math. Proc. Cambridge Philos. Soc.*, **142**(3):557–572, 2007.

[11] Dieudonné, J. *Treatise on Analysis*, Vol. **3**. Academic Press, New York, 1972.

[12] Garetto, C. Fundamental solutions in the Colombeau framework: applications to solvability and regularity theory. *Acta Appl. Math.*, **102**(2-3):281–318, 2008.

[13] Grant, J. D. E., Mayerhofer, E., Steinbauer, R. The wave equation on singular space times. *Commun. Math. Phys.* **285**(2): 399–420, 2009.

[14] Greub, W., Halperin, S., Vanstone, R. *Connections, Curvature, and Cohomology I*, *Pure and Applied Mathematics* **47**. Academic Press, 1972.

[15] Grosser, M. Equivalent representations of distribution spaces on manifolds and automatic continuity. *Preprint*, 2009.

[16] Grosser, M. Tensor valued Colombeau functions on manifolds. In Kaminski, A., Pilipović, S., Oberguggenberger, M., Ed., *Proceedings of the International Conference on Generalized Functions 2007, Bedlewo, Poland*, Banach Center Publications, 2008.

[17] Grosser, M., Farkas, E., Kunzinger, M., Steinbauer, R. On the foundations of nonlinear generalized functions I, II. *Mem. Amer. Math. Soc.*, **153**(729), 2001.

[18] Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer, R. *Geometric Theory of Generalized Functions*, *Mathematics and its Applications* **537**. Kluwer Academic Publishers, Dordrecht, 2001.

[19] Grosser, M., Kunzinger, M., Steinbauer, R., Vickers, J. A global theory of algebras of generalized functions. *Adv. Math.*, **166**:179–206, 2002.

[20] Hörmann, G., Oberguggenberger, M., Pilipović, S. Microlocal hypoellipticity of linear partial differential operators with generalized functions as coefficients. *Trans. Amer. Math. Soc.*, **358**(8):3363–3383, 2006.

[21] Jelínek, J. An intrinsic definition of the Colombeau generalized functions. *Comment. Math. Univ. Carolinae*, **40**:71–95, 1999.

[22] Jelínek, J. On introduction of two diffeomorphism invariant Colombeau algebras. *Comment. Math. Univ. Carolin.*, **45**(4):615–632, 2004.

[23] Jelínek, J. Equality of two diffeomorphism invariant Colombeau algebras. *Comment. Math. Univ. Carolin.*, **45**(4):633–662, 2004.

[24] Keyfitz, B. L., Kranzer. H. C. Spaces of weighted measures for conservation laws with singular shock solutions. *J. Differential Equations* **118**(2), 420–451, 1995.

[25] Kolar, I., Michor, P. W., Slovak, J. *Natural Operations in Differential Geometry*. Springer, Berlin, 1993.

[26] Kriegl, A., Michor, P. W. *The Convenient Setting of Global Analysis*, *Math. Surveys Monogr.* **53**, Amer. Math. Soc., Providence, RI, 1997.

[27] Kunzinger, M., Oberguggenberger, M. Group analysis of differential equations and generalized functions. *SIAM J. Math. Anal.*, **31**(6):1192–1213, 2000.

[28] Kunzinger, M., Steinbauer, R. A note on the Penrose junction conditions. *Class. Quant. Grav.*, **16**:1255–1264, 1999.

[29] Kunzinger, M., Steinbauer, R. Generalized pseudo-Riemannian geometry. *Trans. Amer. Math. Soc.*, **354**(10):4179–4199, 2002.

[30] Kunzinger, M., Steinbauer, R., Vickers, J. Generalized connections and curvature. *Math. Proc. Cambridge Philos. Soc.*, **139**(3):497–521, 2005.

[31] Marsden, J. E. Generalized Hamiltonian mechanics. *Arch. Rat. Mech. Anal.*, **28**(4):323–361, 1968.

[32] Nedeljkov, M., Pilipović, S. Generalized function algebras and PDEs with singularities. A survey. *Zb. Rad. (Beogr.)*, **11**(19):61–120, 2006.

[33] Nedeljkov, M., Pilipović, S., Scarpalézos, D. *The Linear Theory of Colombeau Generalized Functions*, *Pitman Research Notes in Mathematics* **385**. Longman, Harlow, U.K., 1998.

[34] Oberguggenberger, M. *Multiplication of Distributions and Applications to Partial Differential Equations*, *Pitman Research Notes in Mathematics* **259**. Longman, Harlow, U.K., 1992.

[35] Ruzhansky, M. On local and global regularity of Fourier integral operators. *Preprint*, `arXiv:0802.2142v1`, 2008.

[36] Schaefer, H. H. *Topological Vector Spaces.* Springer, New York, 1986.

[37] Schwartz, L. Sur l'impossibilité de la multiplication des distributions. *C. R. Acad. Sci. Paris*, **239**:847–848, 1954.

[38] Steinbauer, R. A geometric approach to full Colombeau algebras. In Kaminski, A., Pilipović, S., Oberguggenberger, M., Ed., *Proceedings of the International Conference on Generalized Functions 2007, Bedlewo, Poland*, Banach Center Publications, 2008.

[39] Steinbauer, R., Vickers, J. A. The use of generalized functions and distributions in general relativity. *Classical Quantum Gravity*, **23**(10):R91–R114, 2006.

[40] Vickers, J. A. Nonlinear generalized functions in general relativity. In Grosser, M., Hörmann, G., Kunzinger, M., Oberguggenberger, M., Ed., *Nonlinear Theory of Generalized Functions*, *CRC Research Notes in Mathematics* **401**, 275–290, CRC Press, Boca Raton, 1999.

[41] Vickers, J. A., Wilson, J. P. Generalized hyperbolicity in conical spacetimes. *Class. Quantum. Grav.*, **17**:1333–1360, 2000.

[42] Vickers, J., Wilson, J. A nonlinear theory of tensor distributions. *ESI-Preprint* **566** (`http://www.esi.ac.at/ESI-Preprints.html`), 1998.